DECOMPOSITION THEOREM FOR BOUNDED UNIFORMLY CONTINUOUS FUNCTIONS ON A GROUP.

By A. W. Knapp.[1]

Introduction. The main theorem of this paper states roughly that the Banach algebra of all bounded left-uniformly continuous functions on a topological group is the direct sum of a space of functions which in a sense vanish at infinity and a space of functions which "oscillate regularly." The first space is any proper self-adjoint closed ideal I maximal with respect to the property of being invariant under left translation. (Such ideals exist by Zorn's Lemma.) The second space is a self-adjoint closed algebra A containing the constants and invariant under left translation.

The decomposition is not canonical, even if I is held fixed. But any A of the type described above that is a direct summand with some such I is a direct summand with every such I, and vice-versa. Each such algebra A contains all almost periodic functions and even the wider class of right distal functions, which are defined in [5].

The set-theoretic union of all the algebras A turns out to be the set of right minimal functions, which are defined intrinsically in Section 3 in terms of limits of their translates. Also in Section 3, right minimal functions are characterized in terms of the ϵ-translation numbers familiar from the theory of almost periodic functions. It is this characterization which gives the sense in which right minimal functions "oscillate regularly."

For studying minimal functions, we shall use shift operators, which are developed in Section 2 and which essentially are just limits of translation operators. This device was suggested partly by the work of Bochner [2]. Shift operators will be used extensively at a later date when we elaborate on the results announced in [5].

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1. Decomposition theorem. Let G be a topological group with identity

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e. If f is a complex-valued function on G, then $\| f \|$ denotes the supremum norm of f and f_g is the left translate of f with $f_g(h) = f(gh)$. Such a function is left-uniformly continuous if $\lim_{g \to e} \| f_g - f \| = 0$. The set of bounded left uniformly continuous functions is denoted LUC; LUC is a self-adjoint (i. e., closed under conjugation) Banach algebra containing the constants and closed under left and right translation.

The theorem to follow is the decomposition theorem. Its proof can be simplified when A_0 is the algebra of constants with the aid of the machinery developed by Ellis in [3] and [4]. The price is the loss of both the elementary nature of the proof and the motivation for considering shift operators.

THEOREM 1-1. *Let I be a proper self-adjoint ideal in* LUC *which is maximal with respect to the property of being invariant under left translation. (Such ideals exist by Zorn's Lemma and are necessarily closed.) Let A_0 be any algebra in* LUC *which*

(a) *contains* 1

(b) *is invariant under left translation and conjugation*

(c) *satisfies* $A_0 \cap I = 0$

(d) *is such that the projection from $A_0 \oplus I$ onto A_0 preserves order ($i.e., a + i \geqq 0$ implies $a \geqq 0$).*

Condition (d) *on A_0 is automatically satisfied if A_0 is closed under uniform limits and satisfies* (a), (b), *and* (c). *Then there exists an algebra A containing A_0 and closed under left translation, conjugation, and uniform limits such that* $\mathrm{LUC} = I \oplus A$.

Proof. (1) We first prove that if A_0 is uniformly closed and satisfies (a), (b), and (c), then A_0 satisfies (d). Thus let A_0 be closed and let $a + i \geqq 0$ with $a \in A$ and $i \in I$. Then $\mathrm{Re}\, a \in A$ and $\mathrm{Re}\, i \in I$, and

$$a + i = \mathrm{Re}\, a + \mathrm{Re}\, i.$$

Since $A_0 \cap I = 0$, a and i must be real. Since A_0 is self-adjoint and uniformly closed, A_0 is closed under absolute value and hence maxima. Since $0 \in A_0$, $\max(a, 0) \in A_0$. If we can show that $\min(a + i, i) \in I$, then the equality

$$a + i = \max(a, 0) + \min(a + i, i)$$

together with the fact that $A_0 \cap I = 0$ shows that $a = \max(a, 0)$ or $a \geqq 0$. Now I can be realized as the set of continuous functions on the maximal ideal space of LUC which vanish on some closed set. On that set, $i = 0$ and $a + i \geqq 0$. Hence on that set, $\min(a + i, i) = 0$, and thus $\min(a + i, i) \in I$.

(2) We now produce A. Partially order by inclusion the set of all algebras in LUC which contain A_0 and which satisfy (a), (b), (c), and (d). Any chain in the set clearly has its union as an upper bound in the set. By Zorn's Lemma take A to be a maximal element.

(3) Next we construct the map T that will be the projection onto A. The sum $A \oplus I$ is a self-adjoint normed algebra which has an identity and is invariant under left translation. Define ρ_0 on $A \oplus I$ to be evaluation of the A part at e. Then ρ_0 is a multiplicative linear functional on $A \oplus I$ which commutes with conjugation. Property (d) implies that ρ_0 is bounded by 1. By uniform continuity extend ρ_0 to ρ_1 defined on the closure of $A \oplus I$ in LUC. Then choose ρ to be a multiplicative linear functional commuting with conjugation which extends ρ_1 to all of LUC; we have $\|\rho\| = 1$. Define T by

$$Tf(g) = \rho(f_g) \text{ for } g \in G.$$

Then T is a homomorphism which commutes with left translation and conjugation. As a map into the space of bounded functions on G, its norm is one. Since

$$\|(Tf)_g - Tf\| = \|Tf_g - Tf\| \leqq \|f_g - f\|,$$

its range is in LUC. Let B be its image in LUC. Then B is a self-adjoint algebra with identity and is invariant under left translation. Therefore B satisfies (a) and (b).

(4) We shall show $\ker T = I$. If $f \in I$, then $f_g \in I$ for every g and hence

$$Tf(g) = \rho(f_g) = \rho_0(f_g) = 0.$$

Thus $\ker T \supseteq I$. In the reverse direction, $\ker T$ is a left-invariant self-adjoint ideal in LUC. By the maximality of I, $\ker T = \text{LUC}$ or $\ker T = I$. The first choice is impossible since $T1 = 1$.

(5) Note that $B \supseteq A$ and that T is the identity on A. In fact, if $f \in A$, then

$$Tf(g) = \rho(f_g) = \rho_0(f_g) = f_g(e) = f(g),$$

so that $Tf = f$, $f \in B$, and T is the identity on A.

(6) We claim that $B \cap I = 0$. In fact, $T^{-1}(B \cap I)$ is a self-adjoint ideal in LUC invariant under left translation, and it contains I by (4). By the maximality of I, $T^{-1}(B \cap I) = \text{LUC}$ or $T^{-1}(B \cap I) = I$. If $T^{-1}(B \cap I) = \text{LUC}$, then

$$B = T(\mathrm{LUC}) = TT^{-1}(B \cap I) = (B \cap I) \cap B = B \cap I$$

and $1 \in I$, a contradiction. Thus $T^{-1}(B \cap I) = I$ and

$$0 = T(I) = T(T^{-1}(B \cap I)) = (B \cap I) \cap B = B \cap I.$$

(7) Next we show that T induces an isometry of LUC/I with B and hence B is closed in LUC. We are to show that

$$\inf_{i \in I} \| f + i \| = \| Tf \|.$$

In one direction

$$\| Tf \| = \| T(f+i) \| \leqq \| T \| \, \| f + i \| = \| f + i \|,$$

and hence $\| Tf \| \leqq \inf_i \| f + i \|$. In the other direction we shall produce an $i \in I$ such that $\| f + i \| = \| Tf \|$. If $\rho_g(f) = \rho(f_g)$, we have

$$\| Tf \| = \sup_{g \in G} | \rho_g(f) |.$$

Let C be the closure in the maximal ideal space of LUC of $\{\rho_g\}$. Then

$$\| Tf \| = \sup_{x \in C} | x(f) |.$$

The functions vanishing on C form a closed self-adjoint left-invariant ideal, and every function in I is annihilated by every ρ_g. By the maximality of I, I is the set of functions in LUC which vanish on C. Extend the restriction of f to C to a continuous function g defined on the entire maximal ideal space of LUC without an increase in norm, and put $i = g - f$. Then $i \in I$ and

$$\| f + i \| = \| g \| = \| f \mid C \| = \sup_{x \in C} | x(f) | = \| Tf \|.$$

Hence B is uniformly closed.

(8) Since B is closed and since B satisfies (a), (b), and (c), step (1) shows that B satisfies (d). By the maximality of A, $B = A$ and A is therefore closed. By (5), T is the identity on A, and A is the image of T since $B = A$. Therefore T is a projection. Its image is A and its kernel, by (4), is I. Thus $\mathrm{LUC} = I \oplus A$, and the proof is complete.

The corollary below was stated and proved by R. Strichartz. It shows that any A that is a direct summand with some I is a direct summand with every I and vice-versa.

COROLLARY 1-2. *If* $\mathrm{LUC} = I \oplus A$ *and* $\mathrm{LUC} = I' \oplus A'$ *are two decompositions given by Theorem* 1-1, *then also* $\mathrm{LUC} = I' \oplus A$.

Proof. Since A and LUC/I are isomorphic, A contains no proper non-zero self-adjoint ideals invariant under left translation. Thus $I' \cap A = 0$ or $I' \cap A = A$. And since 1 is not in I', we must have $I' \cap A = 0$. Therefore by Theorem 1-1 there is a closed A_1 with $A_1 \supseteq A$ and $\mathrm{LUC} = I' \oplus A_1$. Reversing the argument, we can find a closed A_2 with $A_2 \supseteq A_1$ and $\mathrm{LUC} = I \oplus A_2$. Since $A_2 \supseteq A$ and $\mathrm{LUC} = I \oplus A_2 = I \oplus A$, we have $A_2 = A$. Then $A_1 = A$ and $\mathrm{LUC} = I' \oplus A$.

2. Shift operators. The way the projection homomorphism T of LUC onto A in Theorem 1-1 was constructed was to take a certain multiplicative linear functional, translate it around the group, and fit the pieces together. Now any multiplicative linear functional is a weak-star limit of evaluations, and it follows that from any such functional this construction always leads to homomorphisms of LUC which are limits of right-translation operators in an appropriate sense. In this section we introduce the set of all such limiting operators.

We consider nets $\{g_n\}$ in G (i.e., functions on a directed set) such that $\lim_n f(gg_n)$ exists pointwise for every g in G and for every f in LUC. Every convergent net, for instance, has this property. Call two such nets $\{g_n\}$ and $\{h_m\}$ equivalent if

$$\lim_n f(gg_n) = \lim_m f(gh_m)$$

pointwise for every g in G and f in LUC. The operator that sends f into its limiting value for that equivalence class is called the *shift operator* associated to the class. Shift operators are denoted T_α, T_β, etc. Every *right*-translation operator is a shift operator.

Each shift operator is a homomorphism of norm one on LUC, each commutes with conjugation and *left* translation, and each has range in LUC. Since an iterated limit of nets can always be realized as a single limit, the composition of two shift operators is again a shift operator.

The class of shift operators is a set because distinct shift operators are distinct homomorphisms of LUC. A subbase for a topology on this set consists of all sets of the form

$$N(T_\alpha, f, x, \epsilon) = \{T_\beta \mid |T_\beta f(x) - T_\alpha f(x)| < \epsilon\}.$$

In this topology a net $\{T_{\alpha_n}\}$ converges to T_α if, for every f in LUC, the net of functions $\{T_{\alpha_n} f\}$ converges pointwise to $T_\alpha f$.

This topology is the one induced by the multiplicative linear functionals under the construction described in the first paragraph of this section. There is a one-one correspondence between the multiplicative linear functionals and the shift operators defined as follows: If a functional ρ is given, put $T_\alpha f(g) = \rho(f_g)$; any net of evaluations that converges weak-star to ρ is a net in G that defines T_α. On the other hand, if a shift operator T_α is given, put $\rho(f) = T_\alpha f(e)$. We thus get a one-one onto correspondence, and the topology on the set of shift operators is the topology that makes the correspondence a homeomorphism.

Since the weak-star topology on the set of multiplicative linear functionals is compact Hausdorff, so is the topology on the set of shift operators. Moreover, if $\lim T_{\alpha_n} = T_\alpha$, then clearly $\lim T_{\alpha_n} T_\beta = T_\alpha T_\beta$ for fixed T_β. That is, for fixed T_β the map $T_\alpha \to T_\alpha T_\beta$ is continuous. We have proved

THEOREM 2-1. *The set of shift operators is a semigroup under composition and is a compact Hausdorff space. For fixed T_β the map $T_\alpha \to T_\alpha T_\beta$ is continuous.*

Multiplication in the semigroup is not jointly continuous unless every function in LUC is almost periodic. The compactness of the semigroup implies that any net in G has a subnet which defines a shift operator.

3. Minimal functions. We shall be concerned in this section with the set of all algebras A which can occur in Theorem 1-1 if A_0 is taken as the algebra of constants. Evidently any such A is isomorphic to LUC/I, and consequently A contains no proper non-zero self-adjoint left-invariant ideals. This fact has an implication for the maximal ideal space of A, and we begin at this point.

We first recall the definition of a flow and the connection between functions and flows. A *flow* (X, G) for G is a jointly continuous group action of G on a compact Hausdorff space. (We shall suppose the group acts on the left.) If p is a point of X, then for each continuous complex-valued function f on X, the function $\bar{f}$ defined on G as

$$\bar{f}(g) = f(gp)$$

is in LUC, and the set of all such functions $\bar{f}$ for fixed p forms a self-adjoint Banach algebra containing the constants, invariant under left translation, and having maximal ideal space the closure of the orbit of p in X. We shall say $\bar{f}$ arises from the orbit of p. Conversely if such an algebra is given, then the left action of G on itself extends in a nautral way to a jointly continuous action of G on the maximal ideal space of the algebra. These two constructions are inverse to one another if p is chosen to be the evaulation-at-e functional in the maximal space.

We say that $(Y, G) = \pi(X, G)$ is a *subflow* of (X, G) if (X, G) and (Y, G) are flows and if π is a continuous map of X onto Y commuting with G. If $p \in X$, the functions arising from $\pi(p)$ are included in the set of functions arising from p. Conversely an inclusion of algebras of functions induces a map π which exhibits the maximal ideal space of the smaller algebra as a subflow of the maximal ideal space of the larger algebra.

A flow is *minimal* if every orbit is dense, i.e., if it contains no proper non-empty closed invariant subset. A function f in LUC is *right minimal* if for any T_α there is a T_β such that $T_\beta T_\alpha f = f$. If $f \in$ LUC, we denote by A_f the smallest Banach algebra containing f and its left translates, their conjugates, and the constants. If B is a self-adjoint Banach algebra in LUC containing the constants and invariant under left translation, we let $M(B)$ denote its maximal ideal space.

THEOREM 3-1. *A function $f \in$ LUC is right minimal if and only if $M(A_f)$ is a minimal flow. If B is a self-adjoint Banach algebra in* LUC *containing the constants and invariant under left translation, then the following conditions are equivalent:*

(1) *Every function in B is right minimal.*

(2) *B contains no proper non-zero self-adjoint ideals invariant under left translation.*

(3) *$M(B)$ is a minimal flow.*

(4) *$M(A_f)$ is a minimal flow for every $f \in B$.*

Proof. If f is right minimal, then so is every other function in A_f since the T_β corresponding to T_α can be chosen to be the same for all functions in A_f. Therefore the equivalence of (1) and (4) will imply the first statement of the theorem.

(1) $\Rightarrow$ (2). If (2) fails, let f be a function not identically zero in such an ideal, and let ρ be a non-zero multiplicative linear functional on LUC which vanishes on the ideal in B. Define a shift operator T_α by $T_\alpha h(g) = \rho(h_g)$. Then $T_\alpha f = 0$ and no T_β can recover f.

(2) $\Rightarrow$ (3). The closed invariant sets in $M(B)$ are in one-one correspondence with the closed self-adjoint left-invariant ideals in B.

(3) $\Rightarrow$ (4). A subflow of a minimal flow is minimal.

(4) $\Rightarrow$ (1). Let $M(A_f)$ be minimal and let T_α be given. Define a point $x \in M(A_f)$ by $x(h) = T_\alpha h(e)$, and by minimality of $M(A_f)$ find $\{g_n\}$ so that $g_n x$ converges to the evaluation-at-e functional. By compactness of the

set of shift operators, choose a subnet of $\{g_n\}$ which defines some shift operator T_β. Then $T_\beta T_\alpha f = f$.

Functions arising from minimal flows were considered by Auslander and Hahn [1], who proved that (3) and (4) above are equivalent. They showed that for the additive group of reals the functions arising from minimal flows are not closed under addition.

LEMMA 3-2. *The uniform limit of right minimal functions is right minimal.*

Proof. Let $\lim f_n = f$ uniformly with each f_n right minimal, and let T_α be given. Choose T_{β_n} so that $T_{\beta_n} T_\alpha f_n = f_n$, and find a subnet of $\{T_{\beta_n}\}$ which converges, say to T_β. A 3ϵ argument then shows that $T_\beta T_\alpha f = f$.

THEOREM 3-3. *Any algebra A produced by Theorem 1-1 consists entirely of right minimal functions and is a maximal left-invariant self-adjoint algebra of right minimal functions. Conversely any left-invariant self-adjoint algebra of right minimal functions is contained in some A produced by that theorem. Consequently every maximal left-invariant self-adjoint algebra of right minimal functions is such an A, and the set-theoretic union of the A's is exactly the set of all right minimal functions.*

Proof. Since A and LUC$/I$ are isomorphic, A contains no proper non-zero self-adjoint left-invariant ideals and hence, by Theorem 3-1, consists entirely of right minimal functions. Any left-invariant self-adjoint algebra of right minimal functions clearly still consists of right minimal functions when the constants are adjoined. By Lemma 3-2 the uniform closure B of the resulting algebra contains only right minimal functions, and, by (2) of Theorem 3-1, condition (c) of Theorem 1-1 is satisfied by B. Thus B is contained in an algebra A produced by Theorem 1-1.

This result implies that every A produced by the theorem is maximal and that every maximal algebra is an A produced by the theorem. Finally we know that the union of the A's is contained in the set of right minimal functions. The reverse inclusion follows from the fact that if f is right minimal, then A_f is a left-invariant self-adjoint algebra of right minimal functions (Theorem 3-1) and is thus contained in some A.

The algebras A can be characterized quite simply in terms of shift operators. Call T_α minimal if $T_\alpha f$ is right minimal for every $f \in$ LUC; call T_α idempotent if $T_\alpha T_\alpha = T_\alpha$.

THEOREM 3-4. *The decompositions* LUC $= I \oplus A$ *of Theorem* 1-1 *stand* *in one-one correspondence with the minimal idempotent shift operators* T_u, *the correspondence being* $I = \ker T_u$ *and* $A = \text{image } T_u$.

Proof. If $\text{LUC} = I \oplus A$, put $\rho(i + a) = a(e)$ for $i \in I$ and $a \in A$, and define $T_u f(g) = \rho(f_g)$. Then T_u is an idempotent shift operator, and it is minimal by Theorem 3-3. Conversely if T_u is given, then LUC decomposes as $I \oplus A$ and the only problem is to show that I is maximal. But A contains no proper non-zero self-adjoint left-invariant ideals (Theorem 3-1) and hence the same is true of the isomorphic algebra LUC/I. Therefore I is maximal.

COROLLARY 3-5. *A function* f *in* LUC *is right minimal if and only if* $T_u f = f$ *for some minimal idempotent shift operator* T_u.

COROLLARY 3-6. *Any two maximal left-invariant self-adjoint algebras of right minimal functions are isomorphic.*

Proof. Let A_1 and A_2 be given, and put $\text{LUC} = I \oplus A_1 = I \oplus A_2$ by Theorem 3-3 and Corollary 1-2. If T_u and T_v are the minimal idempotents corresponding to these decompositions, then the restriction of T_v to A_1 is the required isomorphism because the restriction of T_u to A_2 is a two-sided inverse.

A comparison of several classes of functions may be helpful. A right almost periodic function is a function in LUC whose right translates form a conditionally compact set in the uniform topology. Right almost periodic functions are also left almost periodic. If $f \in \text{LUC}$, f is right almost automorphic if any net $\{g_n\}$ in G contains a subnet $\{g_{n_j}\}$ such that $\lim f(gg_{n_j})$ exists and

$$\lim_k \lim_j f(gg_{n_k}{}^{-1}g_{n_j}) = f(g).$$

These functions were considered by Bochner [2] and Veech [6]; Veech proved that every right almost automorphic function which is right uniformly continuous is left almost automorphic. If $f \in \text{LUC}$, f is right distal if an equality $T_\alpha T_\beta f = T_\alpha T_\gamma f$ always implies $T_\beta f = T_\gamma f$.

It is easy to see that right almost periodic functions are both right almost automorphic and right distal and that right almost automorphic functions are right minimal. Minimal idempotent shift operators T_u exist by Theorems 1-1 and 3-4, and hence if f is right distal the equality $T_u T_u f = T_u f$ gives $T_u f = f$. Hence right distal functions are right minimal. Using the results of Veech [6] and Bochner [2], one can show also that an almost automorphic right distal function is almost periodic.

We have seen that if f is right distal, then $T_u f = f$ for every minimal idempotent T_u. Veech's results imply that the same is true of all almost automorphic functions. Hence all almost automorphic functions, all right distal functions, and, in particular, all almost periodic functions are contained in every maximal self-adjoint left-invariant algebra of right minimal functions.

A right minimal function on a discrete group need not be left minimal. For an example, consider the action of the discretized unimodular matrix group $SL(2,R)$ on the circle, which is to be thought of as one-dimensional real projective space. This flow is transitive, hence minimal. If p is any point of the circle, then any function arising from the orbit of p is right minimal. On the other hand, each such function has a constant function as the limit of left translates and thus any non-constant such function cannot be left minimal.

We come now to the theorem which tells in what sense right minimal functions oscillate regularly. This result was suggested by H. Furstenberg. An element τ of G is a *right ϵ-translation number* for the function $f \in \mathrm{LUC}$ and for the subset F of G if $|f(g\tau) - f(g)| < \epsilon$ for all $g \in F$. A subset S of G is *right relatively dense* if there exist finitely many group elements $r_1, \cdots, r_n$ such that $G = \bigcup_{i=1}^{n} r_i S$.

For comparison, note that $f \in \mathrm{LUC}$ is right almost periodic if and only if the right ϵ-translation numbers for the set $F = G$ are right relatively dense.

THEOREM 3-7. *If $f \in \mathrm{LUC}$, then f is right minimal if and only if, for each finite subset F of G, the right ϵ-translation numbers for F are right relatively dense.*

Proof. Let f be right minimal and suppose that F is a finite set for which the set $\{\tau\}_F$ of right ϵ-translation numbers is not right relatively dense. To each finite set $E \subseteq G$, associate an element g_E not in $E\{\tau\}_F$; the result is a net in G indexed on the finite subsets of G. Choose a subnet $\{g_m\}$ which defines a shift operator T_α, and find T_β (defined by a net $\{h_n\}$) such that $T_\beta T_\alpha f = f$. Next fix h_n far enough out in the net that

$$|\lim_m f(gh_ng_m) - f(g)| < \epsilon/2$$

for all g in F, and choose an index m beyond the one-point set $\{h_n^{-1}\}$ for which

$$(*) \qquad |f(gh_ng_m) - f(g)| < \epsilon$$

for all g in F. On one hand, g_m was defined not to be in $h_n^{-1}\{\tau\}_F$, and on the other hand $(*)$ gives $h_ng_m \in \{\tau\}_F$. Contradiction.

Conversely, let the right ϵ-translation numbers for each finite set F be right relatively dense and let T_α (defined by $\{g_m\}$) be given. We define a net $\{h_n\}$ on the directed set of pairs (F,ϵ). If (F,ϵ) is given, let S be the set of right ϵ-translation numbers for F and find finitely many r's so that $G=\cup\, rS$. Define $h_{(F,\epsilon)}$ to be an r^{-1} for which $\{g_m\}$ is in rS frequently. We claim that any shift operator T_β defined by a subnet of $\{h_n\}$ satisfies $T_\beta T_\alpha f=f$. It suffices to prove that if $g_0\in G$ and $\delta>0$ are given, then

$$|T_\alpha f(g_0h_{(F,\epsilon)})-f(g_0)|<\delta$$

whenever (F,ϵ) is beyond $(\{g_0\},\delta/2)$. Thus choose such a pair (F,ϵ) and let m_0 be large enough that

$$|T_\alpha f(g_0h_{(F,\epsilon)})-f(g_0h_{(F,\epsilon)}g_m)|<\delta/2$$

whenever $m\geqq m_0$. If S is the set of right ϵ-translation numbers for F, then g_m, by definition, is in $h_{(F,\epsilon)}{}^{-1}S$ frequently. Fix such $m\geqq m_0$. Then $h_{(F,\epsilon)}g_m$ is in S and

$$|f(gh_{(F,\epsilon)}g_m)-f(g)|<\epsilon\leqq\delta/2$$

for all $g\in F$ and in particular for $g=g_0$. Consequently

$$|T_\alpha f(g_0h_{(F,\epsilon)})-f(g_0)|<\delta.$$

4. Connection with the Ellis semigroup. We conclude by showing the connection between shift operators and the Ellis semigroup introduced in [4] and defined as follows. If (X,G) is a flow, we form the closure of G in the space X^X with the product topology. The result is a compact Hausdorff space which is a semigroup under composition and in which the maps $t\to ts$ for fixed s are continuous.

Let B be a self-adjoint Banach algebra in LUC containing the constants and invariant under left translation, and let $M(B)$ be the maximal ideal space of B. We shall use the symbol e interchangeably to denote the identity element of G and the evaluation-at-identity element of $M(B)$.

Relative to B we define an equivalence relation $\sim$ on the set of shift operators. Call $T_\alpha\sim T_\beta$ if $T_\alpha T_\gamma f=T_\beta T_\gamma f$ for all $f\in B$ and for all T_γ. Since the composition of two shift operators is a single shift operator, it follows that $T_\alpha\sim T_\beta$ implies $T_\epsilon T_\alpha\sim T_\epsilon T_\beta$ and $T_\alpha T_\epsilon\sim T_\beta T_\epsilon$. In addition, the relation $\sim$ is a closed relation since multiplication of shift operators is continuous in the first variable. In other words, if $\lim T_{\alpha_n}=T_\alpha$, $\lim T_{\beta_n}=T_\beta$, and $T_{\alpha_n}\sim T_{\beta_n}$ for all n, then $T_\alpha\sim T_\beta$. Therefore the quotient space of the set of shift operators modulo $\sim$ is a compact Hausdorff space with an inherited semigroup structure in which multiplication is continuous in the first variable. Convergence of a net of equivalence classes $[T_{\alpha_n}]$ to $[T_\alpha]$ means that $\lim T_{\alpha_n}T_\gamma f = T_\alpha T_\gamma f$ for all f in A and for all T_γ.

LEMMA 4-1. *Let B be a self-adjoint Banach algebra in* LUC *containing the constants and invariant under left translation, and let T_γ be a shift operator. Then there is a unique point x in $M(B)$ such that*

$$T_\gamma f(g) = f_g(x)$$

for all $g \in G$ and all $f \in B$. Every point x arises in this way from some T_γ.

Proof. The map $f \to T_\gamma f(e)$ is a multiplicative linear functional and hence establishes the existence and uniqueness of x. If x is given, choose a net of evaluations convergent to x and choose a subnet which defines a shift operator T_γ; then T_γ maps onto x.

THEOREM 4-2. *Let B be a self-adjoint Banach algebra in* LUC *containing the constants and invariant under left translation. Then the quotient of the set of shift operators by the equivalence relation induced by B is canonically topologically isomorphic to the Ellis semigroup $\bar{G}$ of $M(B)$.*

Proof. Let t be in $\bar{G}$, and put $Tf(g) = f_g(te)$ for $g \in G$ and $f \in B$. Let $\{g_n\}$ be any net of G-actions in $\bar{G}$ converging to t. Then $\lim f_g(g_n e) = f_g(te)$ by the continuity of f_g on $M(B)$, and hence

$$Tf(g) = \lim f(gg_n)$$

for all $f \in B$. Find a subnet of $\{g_n\}$ which defines a shift operator. We wish to show that different choices for these nets can lead only to equivalent shift operators. Thus let $\lim g_n = t$ in $\bar{G}$ with $\{g_n\}$ defining T_α and let $\lim h_m = t$ in $\bar{G}$ with $\{h_m\}$ defining T_β. Let T_γ be given and choose x as in Lemma 4-1. Then

$$(**) \qquad T_\alpha T_\gamma f(g) = \lim T_\gamma f(gg_n) = \lim f_{gg_n}(x) = \lim f_g(g_n x) = f_g(tx).$$

By symmetry $T_\alpha T_\gamma f = T_\beta T_\gamma f$, and we therefore have a well-defined map from $\bar{G}$ into the quotient space.

This map is one-one. In fact, if t and s are unequal elements of $\bar{G}$ corresponding to T_α and T_β, respectively, let $t(x) \neq s(x)$. If T_γ is chosen according to Lemma 4-1 so that it maps onto x and if f is a function which separates $t(x)$ and $s(x)$, then (**) shows that $T_\alpha T_\gamma(e) \neq T_\beta T_\gamma f(e)$. Hence T_α and T_γ are inequivalent.

The map is onto. For each x in $M(B)$ first choose T_γ by Lemma 4-1 such that $T_\gamma f(g) = f_g(x)$ and then find $t(x)$ by the same lemma such that

$$T_\alpha T_\gamma f(g) = f_g(t(x)).$$

Then t is in $M(A)^{M(A)}$. If $\{g_n\}$ is any net defining T_α, we claim that $\{g_n\}$ converges to t in $M(A)^{M(A)}$ and hence t is in $\bar{G}$. In fact, we have

$$f(t(x)) = T_\alpha T_\gamma f(e) = \lim T_\gamma f(g_n) = \lim f(g_n x).$$

If $\{g_n\}$ does not converge to t, then, by the compactness of $M(A)^{M(A)}$, some subnet converges to $s \neq t$ and gives the contradiction $f(t(x)) = f(s(x))$ for all f and x. Hence t is in $\bar{G}$, and (**) shows that t maps onto the equivalence class of T_α.

Finally we prove continuity. Let $\lim t_n = t$ in $\bar{G}$. We have $\lim f_g(t_n x) = f_g(tx)$ for all g, x, and f. Hence if t_n corresponds to $[T_{\alpha_n}]$ and t corresponds to $[T_\alpha]$, (**) shows that

$$\lim T_{\alpha_n} T_\gamma f(g) = T_\alpha T_\gamma f(g)$$

for all g, T_γ, and f. That is, $\lim [T_{\alpha_n}] = [T_\alpha]$. Thus the map is continuous and must be a homeomorphism.

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