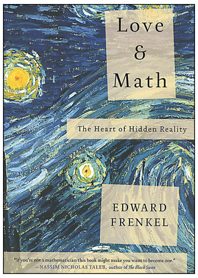

Love and Math: The Heart of Hidden Reality

Reviewed by Anthony W. Knapp

Love and Math: The Heart of Hidden Reality
Edward Frenkel
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Edward Frenkel is professor of mathematics at Berkeley, the 2012 AMS Colloquium Lecturer, and a 1989 émigré from the former Soviet Union. He is also the protagonist Edik in the splendid November 1999 *Notices* article by Mark Saul entitled "Kerosinka: An Episode in the History of Soviet Mathematics." Frenkel's book intends to teach appreciation of portions of mathematics to a general audience, and the unifying theme of his topics is his own mathematical education.

Except for the last of the 18 chapters, a more accurate title for the book would be "Love of Math." The last chapter is more about love than math, and we discuss it separately later in this review.

Raoul Bott once gave a lecture called "Sex and Partial Differential Equations." Come for the sex. Stay for the partial differential equations. The title "Love and Math" is the same idea.

Much of the book is a narrative about Frenkel's own personal experience. If this were all there were to the book, it would be nice, but it might not justify a review in the *Notices*. What sets this book apart is the way in which Frenkel uses his personal story to encourage the reader to see the beauty of some of the mathematics that he has learned. In a seven-page preface, Frenkel says "This book is an invitation to this rich and dazzling world. I wrote it for readers without any background in mathematics." He concludes the preface with this sentence:

> My dream is that all of us will be able to see, appreciate, and marvel at the magic beauty and exquisite harmony of these ideas, formulas, and equations, for this will give so much more meaning to our love for this world and for each other.

Frenkel's Personal Story

Frenkel is a skilled storyteller, and his account of his own experience in the Soviet Union, where he was labeled as of "Jewish nationality" and consequently made to suffer, is gripping. It keeps one's attention and it keeps one wanting to read more. After his failed experience trying to be admitted to Moscow State University, he went to Kerosinka. There he read extensively, learned from first-rate teachers, did mathematics research at a high level, and managed to get some of his work smuggled outside the Soviet Union. The result was that he finished undergraduate work at Kerosinka and was straightway offered a visiting position at Harvard. He tells this story in such an engaging way that one is always rooting for his success. He shows a reverence for various giants who were in the Soviet Union at the time, including I. M. Gelfand and D. B. Fuchs. The account of how he obtained an exit visa is particularly compelling. Once he is in the United States, readers get to see his awe at meeting mathematical giants such as Vladimir Drinfeld and later Edward Witten and Robert Langlands. The reader gets to witness in Chapter 14 the 1990 unmasking of the rector (president) of Moscow University, who had just given a public lecture in Massachusetts and whitewashed that university's discriminatory admissions policies. From there one gets to follow Frenkel's progress through more recent joint work with Witten and up to a collaboration with Langlands and Ngô Bao Châu.

I have to admit that I was dubious when I read the publisher's advertising about the educational aspect of the book. I have seen many lectures and

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books by people with physics backgrounds that contain no mathematics at all—no formulas, no equations, not even any precise statements. Such books tend to suggest that modern physics is really just one great thought experiment, an extension of Einstein's way of thinking about special relativity. *Love and Math* at first sounded to me exactly like that kind of book. I was relieved when I opened *Love and Math* and found that Frenkel was trying not to treat his subject matter this way. He has equations and other mathematical displays, and when his descriptions are more qualitative than that, those descriptions usually are still concrete. The first equation concerns clock arithmetic and appears on page 18, and there are many more equations and mathematical displays starting in Chapter 6. As a kind of compensation for formula-averse readers, he includes a great many pictures and diagrams and tells the reader to "feel free to skip [any formulas] if so desired."

Audience

The equations being as they are, to say that his audience is everyone is an exaggeration. My experience is that the average person in the United States is well below competency at traditional first-year high-school algebra, even though that person may once have had to pass such a course to get a high-school diploma. For one example, I remember a botched discussion on a local television newscast of the meaning of the equation $x^n + y^n = z^n$ after Andrew Wiles announced his breakthrough on Fermat's Last Theorem. As a book that does more than tell Frenkel's own personal story, *Love and Math* is not for someone whose mathematical ability is at the level of those newscasters.

Frenkel really aims at two audiences, one wider than the other. The wider audience consists of people who understand some of the basics of first-year high-school algebra. The narrower audience consists of people with more facility at algebra who are willing to consider a certain amount of abstraction. To write for both audiences at the same time, Frenkel uses the device of lengthy endnotes, encouraging only the interested readers to look at the endnotes. Perhaps he should also have advised the reader that the same zippy pace one might use for reading the narrative parts of the book is not always appropriate for reading the mathematical parts. Anyway, the endnotes occupy 35 pages at the end of the book.

Initial Mathematics

The mathematics begins gently enough with a discussion of symmetry and finite-dimensional representations in Chapter 2. No more mathematics really occurs until Chapter 5, when braid groups are introduced with some degree of detail. Chapter 6 touches on mathematics by alluding to Betti numbers and spectral sequences without really discussing them. It also gives the Fermat equation $x^n + y^n = z^n$ but does not dwell on it.

Chapters 7–9 contain some serious quantitative mathematics, and then there is a break for some narrative. The mathematics resumes in Chapter 14. The topics in Chapters 7–9 quickly advance in level. The mathematical goals of the chapters are respectively to introduce Galois groups and to say something precise yet introductory about the Langlands Program as it was originally conceived [4]. Giving some details but not all, Chapter 7 speaks of number systems—the positive integers, the integers, the rationals, certain algebraic extensions of the rationals, Galois groups, and solutions of polynomial equations by radicals. Frenkel concludes by saying that the Langlands program "ties together the theory of Galois groups and another area of mathematics called harmonic analysis."

Nature of the Langlands Program

It is time in this review to interpolate some remarks about the nature of the Langlands program. The term "Langlands program" had one meaning until roughly 1979 and acquired a much enlarged meaning after that date. The term came into use about the time of A. Borel's Séminaire Bourbaki talk [1] in 1974/75. In the introduction Borel wrote (my translation):

> This lecture tries to give a glimpse of the set of results, problems and conjectures that establish, whether actually or conjecturally, some strong ties between automorphic forms on reductive groups, or representations of such groups, and a general class of Euler products containing many of those that one encounters in number theory and algebraic geometry.
>
> At the present time, several of these conjectures or "questions" appear quite inaccessible in their general form. Rather they define a vast program, elaborated by R. P. Langlands since about 1967, often called the "Langlands philosophy" and already illustrated in a very striking way by the classical or recent results that are behind it, and those that have been obtained since. …
>
> Finally sections 7 and 8 are devoted to the general case. The essential new point is the introduction, by Langlands [in three cited papers] of a group associated to a connected reductive group over a local field, on which are defined L factors of representations of G; also, following a suggestion of H. Jacquet, we shall call it the L-group of G and denote it ${}^LG$. …

After 1979 Langlands and others worked on related matters that expanded the scope of the term "Langlands program." More detail about the period before 1979 and the reasons for the investigation appear in Langlands's Web pages, particularly [4].

Nature of Author's Expository Style

Let us return to Frenkel's book. Some detail about Chapter 8, which is where the Langlands program is introduced, may give the reader a feel for the nature of the author's writing style. It needs to be said that the mathematical level of the topics jumps around. For example, the material on Galois groups and polynomial equations in Chapter 7 is followed by half a page at the beginning of Chapter 8 about proofs by contradiction. Then Chapter 8 states Fermat's Last Theorem and explains briefly how a certain conjecture (Shimura-Taniyama-Weil) about cubic two-variable equations implies the theorem. Frenkel does not say what cubics are allowed in the conjecture, but he indicates in the endnotes, without giving a definition, that the allowable ones are those defining "elliptic curves." He soon gets concrete, working with the specific elliptic curve

$$y^2 + y = x^3 - x^2. \tag{1}$$

He introduces prime numbers and the finite field of integers modulo a prime p. Then he counts the number of finite solution pairs (x, y) of (1) modulo p for some small primes p, observing that the number of solution pairs does not seem easily predictable. He defines a_p by the condition that the number of finite solution pairs is $p - a_p$. Thus he has attached a data set $\{a_p\}$ to the elliptic curve, p running over the primes. Then he considers two examples of sequences definable in terms of a generating function. The first, which is included just for practice with generating functions, is the Fibonacci sequence. The second, studied by M. Eichler in 1954, is the sequence $\{b_p\}$ such that b_p is the coefficient of q^p in

$$q(1-q^2)(1-q^{11})^2(1-q^2)^2(1-q^{22})^2(1-q^3)^2 \times (1-q^{33})^2(1-q^4)^2(1-q^{44})^2 \times \cdots .$$

Frenkel states that the sequences $\{a_p\}$ and $\{b_p\}$ match: $a_p = b_p$ for all primes p. Moreover, he says, this is what the Shimura-Taniyama-Weil Conjecture says for the curve (1). There is no indication why this equality holds, nor could there be in a book of this scope. Frenkel's point is to get across the beauty of the result. The seemingly random integers a_p are thus seen to have a manageable pattern that one could not possibly have guessed.

The Shimura-Taniyama-Weil Conjecture is then tied to the Langlands program on pages 91–92, as follows: Without saying much in the text about what modular forms are, Frenkel explains how the Eichler sequence can be interpreted in terms of modular forms of a certain kind. The endnotes come close to explaining this statement completely. In addition, he says also that the statement of the Shimura-Taniyama-Weil Conjecture is that one can find a modular form of this kind for any elliptic curve. He further says that $\{b_p\}$ can be seen to arise from a two-dimensional representation of the Galois group of an extension of the rationals. Although harder mathematics is coming in later chapters, this is the high point of the concrete mathematics in the book.

In trying to summarize the foregoing, the author goes a little astray at this point and asserts that the correspondence of curves to forms such that the data sets $\{a_p\}$ and $\{b_p\}$ match is one-one. This correspondence is not actually one-one, even if we take into account isomorphisms among elliptic curves. In fact, (1) and the curve

$$y^2 + y = x^3 - x^2 - 10x - 20 \tag{2}$$

are nonisomorphic and correspond to the same modular form.[1] This mistake is not fatal to the book, but it takes the reader's focus off the data sets and is a distraction.

L-Functions

Traditionally the data sets are encoded into certain generating functions called L-functions, which are functions of one complex variable. L-functions and the name for them go back at least to Dirichlet in the nineteenth century. Recall that conjectures about L-functions are at the heart of the Langlands program. One might think of L-functions as of two kinds, arithmetic/geometric and analytic/automorphic. Prototypes for them in the arithmetic/geometric case are the Artin L-functions of Galois representations and the Hasse-Weil L-functions of elliptic curves; in the analytic/automorphic case, prototypes are the Dirichlet L-functions and various L-functions of Hecke. Arithmetic/geometric L-functions contain a great deal of algebraic information, much of it hidden, and analytic/automorphic L-functions tend to have nice properties. The key to unlocking the algebraic information is reciprocity laws, such as the Quadratic Reciprocity Law of Gauss and the Artin Reciprocity Law of E. Artin, which say that certain arithmetic/geometric L-functions coincide with analytic/automorphic L-functions.[2] In some further known cases, the above example of elliptic

[1] *The two curves are isogenous but not isomorphic, according to two of the lines under the heading "11" in the table on page 90 of* [2]. *Their data sets* $\{a_p\}$ *match by Theorem 11.67 of* [3], *for example.*

[2] *The principle still holds if the two are equal except for an elementary factor.*

curves being one of them, a similar reciprocity formula holds, and equality reveals some of the hidden algebraic information.

For example, the Hasse-Weil L-function of an elliptic curve is a specific function of a complex variable s built out of the integers a_p and defined for Re $s > \frac{3}{2}$. Because of the Shimura-Taniyama-Weil Conjecture, this L-function, call it $L(s)$, equals a certain kind of L-function of Hecke, and such functions are known to extend analytically to entire functions of s. Thus $L(1)$ is well defined. The Birch and Swinnerton-Dyer Conjecture, a proof or disproof of which is one of the Clay Millennium Prize Problems, is a precise statement of how the behavior of $L(s)$ near $s = 1$ affects the nature of the rational solution pairs for the curve. Toward rational settling the full conjecture, it is already known that there are only finitely many solution pairs if rational $L(1) \neq 0$ and there are infinitely many solution pairs if $L(1) = 0$ and $L'(1) \neq 0$.

A fond unstated hope of the Langlands program is that every algebraic/geometric L-function can be seen to equal an automorphic L-function except possibly for an elementary factor. L-functions do not appear in *Love and Math*.

Weil's Rosetta Stone

Chapter 9 seeks to fit the Langlands program more fully into a framework first advanced by André Weil, and then it looks at what is missing to include the Langlands program in this framework.[3] In a 1940 letter [6] written from prison to his sister, Weil proposed thinking about three areas of mathematics as written on an imaginary Rosetta stone, one column for each area. The varying columns represent what seems to be the same mathematics, but each is written in its own language. The three languages in Frenkel's terms are number fields,[4] curves over finite fields,[5] and Riemann surfaces.[6] Weil understood some of the entries in each column and sometimes knew how to translate part of one column into part of another. Weil sought to create a dictionary to translate each language into the others. Frenkel wants to add versions of the Langlands program to each column.

Representations of Galois groups of number fields fit tidily into the first column, and the Langlands program conjecturally associates suitable automorphic functions to them. Similarly, as Frenkel says, representations of Galois groups of curves over finite fields fit tidily into the second column, and the Langlands program already associates suitable automorphic functions to them. The question is what to do with the third column (the setting of Riemann surfaces). In endnote 21 for Chapter 9, he gives a careful explanation of how the proper analog of the Galois group of a number field is the fundamental group of the Riemann surface. He can speak easily of fundamental groups because of his detailed treatment of braid groups in Chapter 5. He says that, for a proper analog of automorphic functions in the context of Riemann surfaces, functions are inadequate. He proposes "sheaves" as a suitable generalization of functions, and "automorphic sheaves" will be the objects he uses for harmonic analysis in this setting.

He does not introduce sheaves until Chapter 14, after spending several chapters on his further personal history while touching very briefly on mathematical notions like loop groups and Kac-Moody algebras.

Chapters 14–17

Chapters 14–17 contain the remaining substantive comments on mathematics. They are tough slogging, and they are short on details. Chapter 14 is about sheaves. Some intuition is included, but there is no definition in the text or the endnotes. Nor did I find a single example.

However, the endnotes for Chapter 14 contain a nice discussion of algebraic extensions of finite fields and the Frobenius element of the Galois group of the algebraic closure, and the endnotes go on to illustrate how to compute with the Frobenius element. Chapter 15 is about his and Drinfeld's efforts to merge Frenkel's earlier work on Kac-Moody algebras, which is not actually detailed in the book, with the theory of sheaves. The reader may suppose that this merger can take place and that the result completes the enlargement of Weil's Rosetta stone.

More than half of Chapter 15 is occupied with the screenplay of a conversation between Drinfeld and Frenkel. Part of the screenplay reads as follows:

> DRINFELD writes the symbol ${}^LG$ on the blackboard.
>
> EDWARD
>
> Is the L for Langlands?
>
> DRINFELD
>
> (hint of a smile)
>
> Well, Langlands' original motivation was to understand something called L-functions, so he called this group an L-group …

[3] *Weil's framework predates the Langlands program.*

[4] *Finite extensions of the rationals.*

[5] *Finite algebraic extensions of fields F(x), where F is a finite field.*

[6] *Finite extensions of* $\mathbb{C}(x)$.

The snide inclusion of this exchange is completely uncalled for. We have seen that the term L-function goes back to Dirichlet or earlier and that the name L-group and the notation ${}^L G$ were introduced by Jacquet, not Langlands. Langlands himself had initially used the notation $\widehat{G}$ to indicate what is now known as the L-group.

Chapters 16 and 17 are about quantum duality, string theory, and superstring theory—all in an effort to put them into the context of the Langlands program. In a sense this is ongoing research. To an extent it is also theoretical physics, which has to mesh eventually with the real universe in order to be acceptable. Langlands put comments about this research on his website at [5] on April 6, 2014. He said of Frenkel's articles in this direction,

> These articles are impressive achievements but often freewheeling, so that, although I have studied them with considerable care and learned a great deal from them that I might never have learned from other sources, I find them in a number of respects incomplete or unsatisfactory. …
>
> It [a certain duality that is involved] has to be judged by different criteria. One is whether it is physically relevant. There is, I believe a good deal of scepticism, which, if I am to believe my informants, is experimentally well founded. Although the notions of functoriality and reciprocity have, on the whole, been well received by mathematicians, they have had to surmount some entrenched resistance, perhaps still latent. So I, at least, am uneasy about associating them with vulnerable physical notions. …

Chapter 18

Chapter 18, the last chapter, is completely different from the others. It is unrelated to the theme of the book and simply does not fit. The most charitable explanation that comes to mind for what happened is that inclusion of the chapter was a marketing decision. In any case, including it was a mistake, and I choose to disregard this chapter.

Summary

Thus much of the book is a personal history about the author. This portion is well written and entertaining. Chapters 7–9 present some mathematics that is at once deep and beautiful, and they do so in a way that largely can be appreciated by many readers. The later chapters have nuggets of mathematics that are well done, but not enough to keep the attention of most readers. Most of those later chapters come dangerously close to the content-empty popular physics books that I find merely irritating. Nevertheless, Frenkel's book is a valiant effort at promoting widespread love of mathematics in a wide audience. It could well be that it is actually impossible to write a book of the scope envisioned by Frenkel that achieves this goal fully. If he had not been so ambitious, the result might have been better. For example, stopping after Chapter 9 and including a little more detail might have enabled him to come closer to achieving what he wanted.

Three stars out of five.

Acknowledgment

I am grateful for helpful conversations with Robert Langlands and Martin Krieger.

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[1] A. Borel, Formes automorphes et séries de Dirichlet (d'aprės Langlands), *Séminaire Bourbaki, Exp. 466 (174/75)*, Lect. Notes in Math. 514 (1976), 183–222; *Oeuvres*, III, 374–398.

[2] J. W. Cremona, *Algorithms for Modular Elliptic Curves*, Cambridge University Press, 1992.

[3] A. W. Knapp, *Elliptic Curves*, Princeton University Press, 1992.

[4] R. P. Langlands, "Functoriality," `http://publications.ias.edu/rpl/section/21`. The "editorial comments" and "author's comments" describe early history of the Langlands program and the reasons for its introduction. Various papers of Langlands are accessible from this Web location.

[5] ______, "Beyond endoscopy," `http://publications.ias.edu/rpl/section/25`. Within the essay, see the section called "Message to Peter Sarnak" and especially the subsection "Author's comments (Apr. 6, 2014)."

[6] A. Weil, "Une lettre et un extrait de lettre à Simone Weil," March 26, 1940, *Oeuvres Scientifiques*, I, 244–255; English translation, M. H. Krieger, *Notices Amer. Math. Soc.* **52** (2005), 334–341.