Survey of Irreducible Unitary Representations of Semisimple Lie Groups

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The problem of classifying the irreducible unitary representations of noncompact semisimple Lie groups remains unsolved. For discussion purposes these groups may be regarded as connected closed subgroups of real, complex, or quaternion matrices that are closed under conjugate transpose and have finite center. Special linear groups, orthogonal groups relative to an indefinite Hermitian form (with determinant one and only the identity component used), and symplectic groups give examples.

Early work on this problem, beginning with Bargmann and Gelfand-Naimark in 1947, proceeded group by group. The idea was to work with the Lie algebra action, taking into account how the center of the universal enveloping algebra could act as scalars and how a maximal compact subgroup might act.

Although this approach suggested some features that are important for a general group, the approach was manageable only for certain very special groups. A list of the successes (except that [5] was inadvertently omitted) appears in [13].

To put into context this idea that certain features of the classification for particular groups are important in general, let us review the classification for the 2-by-2 real unimodular group $SL(2,\mathbb{R})$. Most of the irreducible unitary representations for this group fall into three series: the "discrete series," the "principal series," and the "complementary series." There are also three exceptional representations.

The *discrete series* comes in two parts, corresponding to spaces of analytic functions and their complex conjugates. For one part the parametrization is by integers $n \geq 2$, the n^{th} representation taking place in the space of analytic functions in the upper half plane square integrable for $y^{-(n-2)}\,dx\,dy$, with action

$$(\mathcal{D}_n^+(g)f)(z) = (bz+d)^{-n} f\left(\frac{az+c}{bz+d}\right)$$

if g has entries a, b, c, d. These representations are *square integrable* in the sense that they occur as irreducible summands in $L^2(SL(2,\mathbb{R}))$.

The *principal series* comes in two parts. One part is parametrized redundantly by an imaginary parameter iv, the representation taking place in $L^2(\mathbb{R})$ with action

$$(\mathcal{P}^{+,iv}(g)f)(x) = |bx+d|^{-1-iv} f\left(\frac{ax+c}{bx+d}\right).$$

The other part $\mathcal{P}^{-,iv}$ of the principal series takes place in $L^2(\mathbb{R})$ with the same action except that a factor $\operatorname{sgn}(bx + d)$ occurs on the right side. The redundance is given by $\mathcal{P}^{+,iv} \cong \mathcal{P}^{+,-iv}$ and $\mathcal{P}^{-,iv} \cong \mathcal{P}^{-,-iv}$. In addition, $\mathcal{P}^{-,0}$ is reducible and therefore needs to be dropped from the classification.

The "complementary series" is a variation of $\mathcal{P}^{+,iv}$ in which iv is replaced by a real number u with $0 < u < 1$ and the norm is changed from $L^2(\mathbb{R})$ to something else.

The three exceptional representations are the one-dimensional trivial representation and the two irreducible constituents of $\mathcal{P}^{-,0}$. The latter share some algebraic properties with the discrete series, but they are not square integrable. They are called "limits of discrete series," looking somewhat like $\mathcal{D}_1^+$ and its complex conjugate.

The above representations of $SL(2,\mathbb{R})$ already point to the three constructions that are the most important in forming irreducible unitary representations of general semisimple Lie groups. The analytic construction behind the discrete series extends to other groups, with holomorphic functions getting replaced by Dolbeault cohomology (kernel modulo image relative to a $\bar{\partial}$ operator). In turn this construction may be imitated in the language of algebra, and it becomes "cohomological induction," which is a powerful technique for constructing unitary representations under favorable hypotheses on the parameters. The members of the principal series are induced representations from the upper triangular subgroup; this construction generalizes to "parabolic induction," which is a second important technique for constructing unitary representations. The complementary series is obtained from the principal series by a kind of "deformation," and this is the third technique. The three exceptional representations require special discussion, and we shall address these later.

This article will address some successes in the classification problem that are related to the above three techniques: cohomological induction, parabolic induction, and deformation. In addition to [13], some other surveys of the classification problem for irreducible unitary representations are the ones by Clozel [4] and Vogan [19,21]. Two books on representation theory of semisimple Lie groups are [11] and [24].

1 Parabolic induction

For $G = SL(2,\mathbb{R})$, the principal series representations are equivalent with induced representations from the upper triangular subgroup. This group is of the form $P_{\mathfrak{p}} = M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$, where $M_{\mathfrak{p}} = \{\pm 1\}$, $A_{\mathfrak{p}}$ is the positive diagonal subgroup, and $N_{\mathfrak{p}}$ is the upper triangular subgroup with 1's on the diagonal. Corresponding to the parameter $\pm$ indexing the principal series, we associate the trivial or nontrivial character σ of $M_{\mathfrak{p}}$, and corresponding to the parameter iv, we associate the linear functional ν on the Lie algebra of $A_{\mathfrak{p}}$ that picks off iv times the upper left entry. Let ρ pick off the upper left entry itself. The induced representation $U(P_{\mathfrak{p}}, \sigma, \nu)$ takes place in the space of functions

$$\{f : G \to \mathbb{C} \mid f(xman) = e^{-(\nu+\rho)\log a}\sigma(m)^{-1}f(x)\}, \tag{1}$$

the action being $U(P_\mathfrak{p}, \sigma, \nu, g)f(x) = f(g^{-1}x)$. The norm is the L^2 norm of the restriction to the rotation subgroup K. The equivalence with $\mathcal{P}^{\pm,i\nu}$ is given by restricting functions from G to the lower triangular subgroup with 1's on the diagonal and identifying this subgroup with $\mathbb{R}$.

Let us generalize this construction. If G is the group to be studied, then the representation theory of G will require us to work with certain "parabolic subgroups" of the form $P = MAN$ within G, and it is important that the subgroups M be in the class of groups under study. The class of semisimple Lie groups or even of linear semisimple Lie groups does not have this property, and it customary therefore to enlarge the class of groups G. The convention that we shall follow is to work with all reductive Lie groups in the "Harish-Chandra class." This class includes all semisimple Lie groups with finite center and allows also for a small amount of mild disconnectedness and for a nonzero center in the Lie algebra. Such a group G has a vector group as a direct factor, and the complementary factor is denoted ${}^0G$. For the exact definitions and for other group-theoretic terminology, see [12].

We denote the Lie algebra of a Lie group by the corresponding Gothic letter with a subscript 0; the complexification of the Lie algebra is indicated by dropping the subscript. Let K be a maximal compact subgroup of G, and form the corresponding Cartan involutions Θ of G and θ of $\mathfrak{g}_0$. Fix an Iwasawa decomposition $G = KA_\mathfrak{p}N_\mathfrak{p}$ of G.

Let $M_\mathfrak{p}$ be the centralizer of $A_\mathfrak{p}$ in K. The subgroup $P_\mathfrak{p} = M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$, or any of its conjugates, is called a *minimal parabolic subgroup* of G. A *parabolic subgroup* of G is any closed subgroup containing a minimal parabolic subgroup. Each such has a *Langlands decomposition* $P = MAN$, where N is the unipotent radical of P, MA is $P \cap \Theta P$, A is the vector factor of MA, and $M = {}^0(MA)$. The group M is noncompact unless P is minimal. The parabolic subgroup is called *standard* if $P \supseteq P_\mathfrak{p}$. In this case, M contains $M_\mathfrak{p}$, A is contained in $A_\mathfrak{p}$, and N is contained in $N_\mathfrak{p}$. For $G = GL(n, \mathbb{R})$, if $P_\mathfrak{p}$ is the upper triangular subgroup, then the standard P's are the various block upper triangular subgroups.

Let σ be an irreducible unitary representation of M on a Hilbert space V. We obtain representations of G by *parabolic induction* as follows: Let ν be any complex-valued linear functional on $\mathfrak{a}_0$, and let ρ be the linear functional on $\mathfrak{a}_0$ such that conjugation of N by $a \in A$ multiplies Haar measure of N by $e^{2\rho \log a}$. The induced representation $U(P, \sigma, \nu)$ is given by (1) except that the functions take values in V rather than $\mathbb{C}$. If ν is imaginary, then $U(P, \sigma, \nu)$ is unitary.

Parabolic induction is thus a rich source of unitary representations of G. In principle the classification problem makes use of parabolic induction only when σ is a "discrete series" or "limit of discrete series" representation of M. We now define these notions.

2 Discrete series and cohomological induction

The *discrete series* of G consists of all square-integrable irreducible unitary representations of G. These were parametrized by Harish-Chandra [9]. They exist if and only if $\operatorname{rank} G = \operatorname{rank} K$, i.e., K has a Cartan subgroup that is also a Cartan subgroup of G.

We describe Harish-Chandra's parametrization under the assumption that G is connected; the complications from disconnectedness we dismiss as a technical matter. Let T be a Cartan subgroup of K and G, let $\Delta(\mathfrak{g}, \mathfrak{t})$ be the associated system of roots, fix a positive system $\Delta^+(\mathfrak{g}, \mathfrak{t})$, and let δ be half the sum of the positive roots. The discrete series representations corresponding to the choice $\Delta^+(\mathfrak{g}, \mathfrak{t})$ are parametrized by elements Λ in the dual $\mathfrak{t}'$ such that $\Lambda - \delta$ is integral and Λ is dominant and nonsingular. The Λ^{th} representation has Λ in the formula for the global character. Two such representations are equivalent if and only if their Harish-Chandra parameters are related by $\mathrm{Ad}(w)$ for some w in K that normalizes $\mathfrak{t}$.

Schmid, starting in 1967 and following conjectures of Langlands and Kostant that were motivated partly by the Borel-Weil-Bott Theorem for compact groups, showed how to realize the discrete series concretely. An exposition appears in [17]. Let $\mathfrak{n}$ be the sum of the root spaces for the positive roots. Then G/T admits the structure of a complex manifold such that G operates holomorphically and such that the dual $\mathfrak{n}^*$ gives the space of 1-forms of type $d\bar{z}$ at the identity coset. Let Λ be a Harish-Chandra parameter, let $\lambda = \Lambda - \delta$, and let e^λ be the corresponding character of T. Let $\mathcal{B}_\lambda$ be the associated holomorphic line bundle $G \times_T \mathbb{C}_\lambda^\#$ over G/T, where T acts on $\mathbb{C}_\lambda^\# \cong \mathbb{C}$ by $e^{\lambda+2\delta}$.

For $SL(2, \mathbb{R})$, $\mathcal{B}_\lambda$ can admit nonzero holomorphic sections, but this is not true in general. Instead one introduces a $\bar{\partial}$ operator, which carries smooth sections of $G \times_T ((\wedge^q \mathfrak{n})^* \otimes \mathbb{C}_\lambda^\#)$ to smooth sections of $G \times_T ((\wedge^{q+1} \mathfrak{n})^* \otimes \mathbb{C}_\lambda^\#)$. The group G acts on these spaces of sections, and the kernel modulo image gives, after completion, a unitary representation $H^{0,q}(G/T, \mathbb{C}_\lambda^\#)$ of G.

Theorem 2.1 *Let $\lambda + \delta$ be a Harish-Chandra parameter, and let $S = \dim(\mathfrak{n} \cap \mathfrak{k})$. The representation of G in $H^{0,q}(G/T, \mathbb{C}_\lambda^\#)$ is 0 if $q \neq S$, and it is the discrete series representation with Harish-Chandra parameter $\lambda + \delta$ if $q = S$.*

The cohomology spaces $H^{0,q}(G/T, \mathbb{C}_\lambda^\#)$ are difficult to work with directly. In an effort to generalize their construction, Zuckerman in 1978 considered a manageable algebraic analog that he was able to generalize well. To understand Zuckerman's theory, it is necessary first to say how group representations can be treated algebraically.

A representation of G in a Hilbert space is *admissible* if, when restricted to the compact group K, it contains each K type (i.e., class of irreducible representations of K) only with finite multiplicity. Irreducible unitary representations all have this property, and so do representations of G obtained by parabolic induction from an irreducible unitary representation. To any finitely generated admissible representation of G on a Hilbert space V, we can associate a so-called $(\mathfrak{g}, K)$ module that is finitely generated and admissible. The vector space is the space V_K of all K finite vectors. On this space, there is a natural K representation. The space is left stable by the action of $\mathfrak{g}_0$ and hence also by the action of the complexification $\mathfrak{g}$. The actions of $\mathfrak{g}$ and K are suitably compatible, and the result is a $(\mathfrak{g}, K)$ module. The closed G invariant subspaces of V correspond to the $(\mathfrak{g}, K)$ invariant subspaces of V_K; one passes from closed invariant subspaces of V to invariant subspaces of V_K by intersecting with V_K, and one passes in the reverse direction by taking the closure.

Every finitely generated admissible $(\mathfrak{g}, K)$ module arises in this way from some admissible representation of G.

Zuckerman's idea was to construct what ought to be the underlying $(\mathfrak{g}, K)$ modules of his cohomology spaces by using Taylor coefficients at the identity. A closed $(0, q)$ form has Taylor coefficients at 1 corresponding to all the derivatives at 1 of a certain kind of function on G with values in $(\wedge^q \mathfrak{n})^* \otimes \mathbb{C}_\lambda^\#$. If $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$, this system of derivatives may be viewed as a member of $\mathrm{Hom}(U(\mathfrak{g}), (\wedge^q \mathfrak{n})^* \otimes \mathbb{C}_\lambda^\#)$. Working with constructions in homological algebra, Zuckerman was led to form the $(\mathfrak{g}, T)$ modules $\mathrm{Hom}_{\mathfrak{t}+\mathfrak{n},T}(U(\mathfrak{g}), \mathbb{C}_\lambda^\#)_T$, where $\mathfrak{n}$ acts on $\mathbb{C}_\lambda^\#$ by 0 and where T denotes the T finite subspace. Let Γ be the operation of passing to the subspace of all elements for which the action of $\mathfrak{k}_0$ globalizes to K; Γ converts $(\mathfrak{g}, T)$ modules into $(\mathfrak{g}, K)$ modules. Let Γ^q denote the q^{th} right derived functor of Γ in the sense of homological algebra. Zuckerman used $\Gamma^q(\mathrm{Hom}_{\mathfrak{t}+\mathfrak{n}}(U(\mathfrak{g}), \mathbb{C}_\lambda^\#)_T)$ as an algebraic analog of $H^{0,q}(G/T, \mathbb{C}_\lambda^\#)$, and he sketched a proof that his modules were 0 when $q \neq S$ and were the underlying $(\mathfrak{g}, K)$ modules of the appropriate discrete series when $q = S$.

One way, not the original way, of defining limits of discrete series is in this setting. With λ still integral, suppose that $\lambda + \delta$ is dominant but singular. If $\Gamma^S(\mathrm{Hom}_{\mathfrak{t}+\mathfrak{n}}(U(\mathfrak{g}), \mathbb{C}_\lambda^\#)_T)$ is not 0, it can be shown to be the underlying $(\mathfrak{g}, K)$ module of an irreducible unitary representation of G, and such a representation is called a *limit of discrete series*. The condition for this representation to be nonzero is that λ plus the sum of the noncompact positive roots be dominant for K.

General *cohomological induction* is as follows: In place of $\mathfrak{t} \oplus \mathfrak{n}$, one uses a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ that is suitably θ stable; here $\mathfrak{l} = \mathfrak{q} \cap \bar{\mathfrak{q}}$ is the Levi factor and $\mathfrak{u}$ is the nilpotent radical. In place of T one uses the subgroup L of G given as the normalizer of $\mathfrak{q}$ in G; L has complexified Lie algebra $\mathfrak{l}$ and need not be compact. We begin with a $(\mathfrak{l}, L \cap K)$ module Z, shift it to $Z^\# = Z \otimes \wedge^{\text{top}} \mathfrak{u}$, and extend $Z^\#$ to a $(\mathfrak{q}, L \cap K)$ module by having $\mathfrak{u}$ act by 0. Then $\mathrm{Hom}_{\mathfrak{q}, L \cap K}(U(\mathfrak{g}), Z^\#)_{L \cap K}$ is a $(\mathfrak{g}, L \cap K)$ module. Applying Γ^q gives a $(\mathfrak{g}, K)$ module denoted $\mathcal{R}^q(Z)$. This construction is meaningful for arbitrary G in the Harish-Chandra class if Γ is defined carefully enough. An extensive theory concerning this construction was developed by Vogan [18], and an important early theorem was proved by Enright and Wallach [8]. For an exposition of cohomological induction, see [14]. When Z is finite-dimensional, Wong [25] has shown that the analytic construction using Dolbeault cohomology always results in representations in Hausdorff spaces and that cohomological induction gives the underlying $(\mathfrak{g}, K)$ modules.

Let us describe the effect of cohomological induction on irreducibility and unitarity. For simplicity we assume G is connected. First let us introduce terminology concerning unitarity. A $(\mathfrak{g}, K)$ module is *infinitesimally unitary* if it admits a Hermitian inner product preserved by K such that $\mathfrak{g}_0$ acts by skew-Hermitian transformations. Any finitely generated admissible representation of G that is unitary has an infinitesimally unitary underlying $(\mathfrak{g}, K)$ module, and conversely an admissible finitely generated infinitesimally unitary $(\mathfrak{g}, K)$ module underlies a unitary representation of G; if the $(\mathfrak{g}, K)$ module is irreducible, the unitary representation of G is unique up to unitary equivalence.

In any irreducible $(\mathfrak{g}, K)$ module, the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by scalars, and these scalars yield a homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$. If $\mathfrak{h}$ is any Cartan subalgebra of $\mathfrak{g}$, the "Harish-Chandra isomorphism" exhibits $Z(\mathfrak{g})$ as isomorphic to the Weyl-group invariants in $U(\mathfrak{h})$, and then χ may be identified, up to a Weyl-group transform, with a member of the dual $\mathfrak{h}^*$.

In our setting let $\mathfrak{h}_0$ be any θ stable Cartan subalgebra of $\mathfrak{l}_0$; then $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{g}_0$. Choose a positive system $\Delta^+(\mathfrak{l}, \mathfrak{h})$ from the roots of $\mathfrak{l}$. The union of $\Delta^+(\mathfrak{l}, \mathfrak{h})$ and the set $\Delta(\mathfrak{u})$ of roots contributing to $\mathfrak{u}$ is a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ for $\mathfrak{g}$. Let $\delta(\mathfrak{u})$ be half the sum of the members of $\Delta(\mathfrak{u})$. If Z is an irreducible $(\mathfrak{l}, L \cap K)$ module with infinitesimal character λ relative to $\mathfrak{h}$, we say that λ is in the *weakly good range* if

$$\mathrm{Re}\langle \lambda + \delta(\mathfrak{u}), \alpha \rangle \geq 0 \qquad \text{for all } \alpha \in \Delta(\mathfrak{u}). \tag{2}$$

We say that λ is in the *good range* if strict inequality holds in (2).

Theorem 2.2 *With G connected, let Z be an irreducible admissible $(\mathfrak{l}, L \cap K)$ module with infinitesimal character λ. Define $S = \dim(\mathfrak{u} \cap \mathfrak{k})$. Then $\mathcal{R}^q(Z)$ is an admissible $(\mathfrak{g}, K)$ module with infinitesimal character $\lambda + \delta(\mathfrak{u})$ for all q. If λ is in the weakly good range, then*

(a) $\mathcal{R}^q(Z) = 0$ for $q \neq S$,

(b) $\mathcal{R}^S(Z)$ is irreducible or 0, and

(c) Z infinitesimally unitary implies $\mathcal{R}^S(Z)$ infinitesimally unitary.

If λ is actually in the good range, then

(d) $\mathcal{R}^S(Z)$ is not 0 and

(e) $\mathcal{R}^S(Z)$ infinitesimally unitary implies Z infinitesimally unitary.

Parts (c), (d), and (e) of Theorem 2.2 are due to Vogan [20]. Wallach [24] gave a shorter proof of (c).

3 Langlands classification and deformation

The Langlands classification, obtained in 1973–76 and appearing in [16,15], classifies all irreducible admissible representations up to "infinitesimal equivalence." Two finitely generated admissible representations of G are *infinitesimally equivalent* if their underlying $(\mathfrak{g}, K)$ modules are algebraically equivalent. Since every irreducible unitary representation of G is admissible, the Langlands classification reduces the classification problem for irreducible unitary representations to the question of deciding which of some specific representations admit invariant inner products.

In more detail the proof of the Langlands classification shows which irreducible admissible representations have $(\mathfrak{g}, K)$ modules that admit nonzero invariant Hermitian forms. When such a form exists, it is unique up to a scalar factor, and the question is whether this form is semidefinite. Historically, a direct approach to unitarity by seeing which of these Hermitian forms were semidefinite resulted in some further successes in finishing the classification of irreducible unitary representations for particular groups, and a list of the groups in question appears in [13].

In stating the Langlands classification, we shall not insist on having representations appear only once; consequently the classification will be easier to state. The classification makes use of the parabolically induced representations $U(P,\sigma,\nu)$ of §2, where $P = MAN$ is a parabolic subgroup of G and σ is a discrete series or limit of discrete series of M. It is enough to use a complete system of nonconjugate parabolic subgroups, and the standard ones will suffice. Since M has discrete series if and only if $\operatorname{rank} M = \operatorname{rank}(K \cap M)$, we may assume this equal-rank condition on M; in this case P is called *cuspidal*. When $G = GL(n,\mathbb{R})$, the cuspidal standard parabolic subgroups are the block upper triangular subgroups whose blocks are all of size ≤ 2.

The nonzero simultaneous eigenvalues of $\operatorname{ad}\mathfrak{a}_0$ on $\mathfrak{g}_0$ are called $\mathfrak{a}$ roots, and the ones that correspond to $\mathfrak{n}_0$ are called *positive*. The *positive Weyl chamber* of $\mathfrak{a}_0$ is the subset where the positive $\mathfrak{a}$ roots are simultaneously positive. It determines a positive Weyl chamber in the dual $\mathfrak{a}_0^*$.

Let $W(G,A)$ be the quotient of the normalizer of A by the centralizer. Elements of $W(G,A)$ act on ν's and on classes of σ's, and we let $W_{\sigma,\nu}$ be the subgroup fixing ν and the class of σ. We omit the definition of a subgroup $W'_{\sigma,\nu}$ of $W_{\sigma,\nu}$ that is defined in [13] other than to say that $W'_{\sigma,\nu} = W_{\sigma,\nu}$ with ν imaginary implies that $U(P,\sigma,\nu)$ is irreducible.

Theorem 3.1 [16,15] *Let $P = MAN$ be a cuspidal standard parabolic subgroup of G, let σ be a discrete series or limit of discrete series representation of M, and let ν be a complex-valued linear functional on $\mathfrak{a}_0$ with $\operatorname{Re}\nu$ in the closed positive Weyl chamber. Suppose that $W'_{\sigma,\nu} = W_{\sigma,\nu}$. Then the induced representation $U(P,\sigma,\nu)$ has a unique irreducible quotient $J(P,\sigma,\nu)$, and every irreducible admissible representation of G is of the form $J(P,\sigma,\nu)$ for some such triple (P,σ,ν).*

Because of Theorem 3.1 the classification problem for irreducible unitary representations would be solved if we could decide which representations $J(P,\sigma,\nu)$ are infinitesimally unitary. To describe what needs to be done, we shall use an explicit intertwining operator $\sigma(w)A_P(w,\sigma,\nu)$ whose exact definition will not concern us.

Theorem 3.2 [13] *Let (P,σ,ν) be such that the irreducible admissible representation $J(P,\sigma,\nu)$ is defined. Then $J(P,\sigma,\nu)$ is infinitesimally unitary if and only if*

(i) there exists w in $W(G,A)$ such that $w^2 = 1$, $w\sigma \cong \sigma$, and $w\nu = -\bar{\nu}$, and

(ii) the operator $\sigma(w)A_P(w,\sigma,\nu)$, when normalized to be pole-free and not identically zero, is positive or negative semidefinite.

If $J(P,\sigma,\nu)$ is infinitesimally unitary, then every w satisfying (i) is such that the normalized operator is positive or negative semidefinite.

The normalized operator may be viewed as varying holomorphically, the representation $J(P, \sigma, \nu)$ is always the image of the normalized operator, and the invariant form is given relative to $L^2(K)$ by the normalized operator. When the operator is scalar for ν imaginary, it remains definite as long as the symmetry condition (i) is satisfied and $U(P, \sigma, \nu)$ is not reducible. It is in this sense that the invariant inner product is sometimes obtained by *deformation* from $L^2(K)$. This style of argument accounts for the unitarity of the complementary series of $SL(2, \mathbb{R})$.

It follows from Theorem 3.2 that the set of real ν's giving unitarity for a fixed σ is a kind of bounded polygonal simplex. A picture in [13] gives an example. More complicated versions of deformation often establish much of the unitarity. But the set where unitarity holds is not necessarily connected, and deformation cannot handle everything. Thus to decide unitarity of certain $J(P, \sigma, \nu)$, it is often helpful to have a second description of the representation.

One important fact is that there is an interaction between cohomological induction and the Langlands classification. Namely cohomological induction carries $U(P_L, \sigma_L, \nu_L)$ for L to $U(P, \sigma, \nu)$ for G when the parameters are matched properly. There are some provisos: The functor $\mathcal{R}^S$ is to be replaced by a kind of predual $\mathcal{L}_S$ discussed in [14], cohomological induction is to be normalized with a half sum of positive roots, A and ν are to be the same for L as for G, the positive Weyl chambers must be suitably consistent, and $\operatorname{Re}\nu$ is to be dominant.

4 Results

- When $\dim A = 0$, P equals G, and the representations $J(P, \sigma, \nu)$ are the discrete series and limits. These are all unitary [15].
- It is enough to decide unitarity for $J(P, \sigma, \nu)$ with ν real. This is due to Vogan. See [11], Theorem 16.10.
- For fixed G and P, unitarity of $J(P, \sigma, \nu)$ needs to be settled only for finitely many σ. In fact, we refer to the end of the previous section. The weakly good range for fixed σ depends only on $\operatorname{Re}\nu$ and is a convex polygonal set centered at the origin. When the parameter of σ is suitably large, the weakly good range includes all points ν where unitarity is possible. Consequently Theorem 2.2 implies that there are only finitely many σ's for which unitarity cannot be decided by referring to some proper subgroup L. This result is due to Vogan [20]. See Chapter XI of [14] for further information.
- The papers [23], [6], and [10] completely settle unitarity for cohomological induction outside the weakly good range when $S = 0$. Later [7] investigated some other cases outside the weakly good range. A certain amount of current research seeks to decide unitarity in cases that [7] could not handle.
- When $\dim A = 1$, the unitarity problem is completely solved. Except for split F_4 and G_2, this was proved in [1]. Split F_4 was handled by Zhonghu Chen, and split G_2 was handled by Vogan. The real ν with $J(P, \sigma, \nu)$ unitary form either an interval or an interval with one extra point. Deformation proves unitarity for the interval, and the extra points are

handled by cohomological induction.

• Vogan [22] settled the classification problem for $GL(n)$ over the reals, the complex numbers, and the quaternions.

• Barbasch [2] settled the classification problem for the classical orthogonal and symplectic groups. This work relies somewhat on [3].

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