CHAPTER IX

Induced Representations and Branching Theorems

Abstract. The definition of unitary representation of a compact group extends to the case that the vector space is replaced by an infinite-dimensional Hilbert space, provided care is taken to incorporate a suitable notion of continuity. The theorem is that each unitary representation of a compact group G splits as the orthogonal sum of finite-dimensional irreducible invariant subspaces. These invariant subspaces may be grouped according to the equivalence class of the irreducible representation, and there is an explicit formula for the orthogonal projection on the closure of the sum of all the spaces of a given type. As a result of this formula, one can speak of the multiplicity of each irreducible representation in the given representation.

The left-regular and right-regular representations of G on $L^2(G)$ are examples of unitary representations. So is the left-regular representation of G on $L^2(G/H)$ for any closed subgroup H. More generally, if H is a closed subgroup and σ is a unitary representation of H, the induced representation of σ from H to G is an example. If σ is irreducible, Frobenius reciprocity says that the multiplicity of any irreducible representation τ of G in the induced representation equals the multiplicity of σ in the restriction of τ to H.

Branching theorems give multiplicities of irreducible representations of H in the restriction of irreducible representations of G. Three classical branching theorems deal with passing from $U(n)$ to $U(n-1)$, from $SO(n)$ to $SO(n-1)$, and from $Sp(n)$ to $Sp(n-1)$. These may all be derived from Kostant's Branching Theorem, which gives a formula for multiplicities when passing from a compact connected Lie group to a closed connected subgroup. Under a favorable hypothesis the Kostant formula expresses each multiplicity as an alternating sum of values of a certain partition function.

Some further branching theorems of interest are those for which G/H is a compact symmetric space in the sense that H is the identity component of the group of fixed elements under an involution of G. Helgason's Theorem translates into a theorem in this setting for the case of the trivial representation of H by means of Riemannian duality. An important example of a compact symmetric space is $(G \times G)/\operatorname{diag} G$; a branching theorem for this situation tells how the tensor product of two irreducible representations of G decomposes.

A cancellation-free combinatorial algorithm for decomposing tensor products for the unitary group $U(n)$ is of great utility. It leads to branching theorems for the compact symmetric spaces $U(n)/SO(n)$ and $U(2n)/Sp(n)$. In turn the first of these branching theorems helps in understanding branching for the compact symmetric space $SO(n+m)/(SO(n) \times SO(m))$.

Iteration of branching theorems for compact symmetric spaces permits analysis of some complicated induced representations. Of special note is $L^2(K/(K \cap M_0))$ when G is a reductive Lie group, K is the fixed group under the global Cartan involution, and MAN is the Langlands decomposition of any maximal parabolic subgroup.