APPENDIX B

Lie's Third Theorem

Abstract. A finite-dimensional real Lie algebra is the semidirect product of a semisimple subalgebra and the solvable radical, according to the Levi decomposition. As a consequence of this theorem and the correspondence between semidirect products of Lie algebras and semidirect products of simply connected analytic groups, every finite-dimensional real Lie algebra is the Lie algebra of an analytic group. This is Lie's Third Theorem.

Ado's Theorem says that every finite-dimensional real Lie algebra admits a one-one finite-dimensional representation on a complex vector space. This result sharpens Lie's Third Theorem, saying that every real Lie algebra is the Lie algebra of an analytic group of matrices.

The Campbell–Baker–Hausdorff Formula expresses the multiplication rule near the identity in an analytic group in terms of the linear operations and bracket multiplication within the Lie algebra. Thus it tells constructively how to pass from a finite-dimensional real Lie algebra to the multiplication rule for the corresponding analytic group in a neighborhood of the identity.

1. Levi Decomposition

Chapter I omits several important theorems about general finite-dimensional Lie algebras over $\mathbb{R}$ related to the realization of Lie groups, and those results appear in this appendix. They were omitted from Chapter I partly because in this treatment they use a result about semisimple Lie algebras that was not proved until Chapter V. One of the results in this appendix uses also some material from Chapter III.

Lemma B.1. Let φ be an $\mathbb{R}$ linear representation of the real semisimple Lie algebra $\mathfrak{g}$ on a finite-dimensional real vector space V. Then V is completely reducible in the sense that there exist invariant subspaces $U_1, \ldots, U_r$ of V such that $V = U_1 \oplus \cdots \oplus U_r$ and such that the restriction of the representation to each U_i is irreducible.

PROOF. It is enough to prove that any invariant subspace U of V has an invariant complement W. By Theorem 5.29, there exists an invariant