INTRODUCTION

This Introduction provides historical background and motivation for cohomological induction and gives an overview of the five main theorems. The section of Notes at the end of the book points to expositions where more detail can be found, and it gives references for the results that are cited. The Introduction is not logically necessary for the remainder of the book, and it occasionally uses mathematics that is not otherwise a prerequisite for the book.

The first part of the Introduction tells the sense in which representation theory of a semisimple Lie group G with finite center reduces to the study of "$(\mathfrak{g}, K)$ modules," and it describes the early constructions of infinite-dimensional group representations. One of the constructions is from complex analysis and produces representations in spaces of Dolbeault cohomology sections over a complex homogeneous space of G. This construction is expected to lead often to irreducible unitary representations. Passage to Taylor coefficients leads to an algebraic analog of this construction and to the definition of the left-exact Zuckerman functor Γ.

Cohomological induction involves more than the Zuckerman functor and its derived functors; it involves also the passage from a parabolic subalgebra of $\mathfrak{g}$ to $\mathfrak{g}$ itself. The original construction of Zuckerman's does not lend itself naturally to the introduction of invariant Hermitian forms, and for this reason a right-exact version of the Zuckerman functor, known as the Bernstein functor Π, is introduced. The definition of Π depends on introduction of a "Hecke algebra" $R(\mathfrak{g}, K)$, and Π is then given as a change of rings.

1. Origins of Algebraic Representation Theory

Harish-Chandra's first work in representation theory used particular representations of specific noncompact groups to address problems in mathematical physics. In the late 1940s, long after Élie Cartan and Hermann Weyl had completed their development of the representation theory of compact connected Lie groups, Harish-Chandra turned his attention to compact groups and reworked the Cartan-Weyl theory in his own way. Introducing what are now known as Verma modules, he gave a uniform, completely algebraic construction of the irreducible representations of such groups. (Chevalley independently gave a different such algebraic construction. See the Notes for details.)

Motivated by a question in Mautner [1950] of whether connected semisimple Lie groups are of "type I," and perhaps emboldened by the success with finite-dimensional representations, Harish-Chandra began an algebraic treatment of the infinite-dimensional representations of noncompact groups, concentrating largely on connected real semisimple groups. Although he initially allowed arbitrary connected semisimple groups, he eventually imposed the hypothesis that the groups have finite center, and we shall concentrate on that case. Let G be such a group.

Harish-Chandra worked with representations of G on a complex Banach space V, with the continuity property that the action $G \times V \to V$ is continuous. His early goal was to strip away any need for real or complex analysis with such representations and to handle them purely in terms of algebra.

If π is a continuous representation of G on the Banach space V, then the norms of the operators $\pi(x)$ are uniformly bounded for x in compact neighborhoods of 1, and we can average π by L^1 functions of compact support:

$$\pi(f)v = \int_G f(x)\pi(x)v\,dx \qquad \text{for } v \in V \text{ and } f \text{ compactly supported in } L^1(G).$$

Here dx is a Haar measure on G and is two-sided invariant because G is semisimple. The integral may be interpreted either as a vector-valued "Bochner integral" or as an ordinary Lebesgue integral for the value of every continuous linear functional $\langle\,\cdot\,, l\rangle$ on $\pi(f)v$, namely $\langle \pi(f)v, l\rangle = \int_G f(x)\langle \pi(x)v, l\rangle\,dx$.

We say that $v \in V$ is a C^∞ **vector** if $x \mapsto \pi(x)v$ is a C^∞ function from G into V. Gårding observed that the subspace $C^\infty(V)$ of C^∞ vectors is dense in V. In fact, if v is in V and if $f_n \geq 0$ is a sequence of compactly supported C^∞ functions on G of integral 1 and with support shrinking to $\{1\}$, then $\pi(f_n)v$ is a C^∞ vector for each n and $\pi(f_n)v \to v$.

Let $\mathfrak{g}_0$ be the Lie algebra of G. We can make $\mathfrak{g}_0$ act on the space of C^∞ vectors by the definition

$$\pi(X)v = \frac{d}{dt}\pi(\exp tX)v|_{t=0} \qquad \text{for } X \in \mathfrak{g}_0,\ v \in C^\infty(V),$$

and one can check that π becomes a representation of $\mathfrak{g}_0$:

$$\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X) \qquad \text{on } C^\infty(V).$$

As a consequence if $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ denotes the complexification of $\mathfrak{g}_0$ and if $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$, then π extends uniquely from a linear map $\pi : \mathfrak{g}_0 \to \mathrm{End}_{\mathbb{C}}(C^\infty(V))$ to a complex-linear algebra homomorphism $\pi : U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(C^\infty(V))$ sending 1 to 1. In this way $C^\infty(V)$ becomes a $U(\mathfrak{g})$ module.

Under this construction a group representation leads to a $U(\mathfrak{g})$ module in such a way that closed G invariant subspaces W yield $U(\mathfrak{g})$ submodules

$C^\infty(W)$. This correspondence, however, is inadequate as a reduction to algebra of the analytic aspects of representation theory. For one thing, simple examples show that the closure of a $U(\mathfrak{g})$ submodule of $C^\infty(V)$ need not be G invariant. For another, $U(\mathfrak{g})$ has countable vector-space dimension while $C^\infty(V)$ typically has uncountable dimension; thus $C^\infty(V)$ is usually not close to being irreducible (i.e., simple as a $U(\mathfrak{g})$ module) even if V is irreducible.

Harish-Chandra rectified the first problem by using the $U(\mathfrak{g})$ invariant subspace $C^\omega(V) \subseteq C^\infty(V)$ of **analytic vectors**, the subspace of v's for which $x \mapsto \pi(x)v$ is real analytic on G. It is not hard to check that the closure in V of a $U(\mathfrak{g})$ invariant subspace of $C^\omega(V)$ is G invariant. It is still true that $C^\omega(V)$ is dense in V, but this fact is much more difficult to prove than its C^∞ analog and we state it as a theorem.

Theorem 0.1 (Harish-Chandra). If π is a continuous representation of G on a Banach space V, then the subspace $C^\omega(V)$ of analytic vectors is dense in V.

To deal with the problem that $C^\infty(V)$ and even $C^\omega(V)$ are too large, Harish-Chandra made use of a maximal compact subgroup K of the semisimple group G. We say that $v \in V$ is K **finite** if $\pi(K)v$ spans a finite-dimensional space. The subspace of K finite vectors breaks into a (possibly infinite) direct sum of finite-dimensional subspaces on which K operates irreducibly. The subspace V_K of K finite vectors is dense in V, by an averaging argument similar to the proof that $C^\infty(V)$ is dense. Actually even the subspace $C^\omega(V)_K$ of K finite vectors in $C^\omega(V)$ is dense and is a direct sum of finite-dimensional subspaces on which K operates irreducibly.

It is a simple matter to show that $C^\omega(V)_K$ and $C^\infty(V)_K$ are $U(\mathfrak{g})$ submodules of $C^\infty(V)$. Thus $C^\infty(V)_K$ and $C^\omega(V)_K$ are both $U(\mathfrak{g})$ modules and representation spaces for K, and the $U(\mathfrak{g})$ and K structures evidently satisfy certain compatibility conditions. Following terminology introduced by Lepowsky [1973], we call $C^\infty(V)_K$ with its $U(\mathfrak{g})$ and K structures the **underlying** $(\mathfrak{g}, K)$ **module** of V. (See Chapter I for the precise definition of $(\mathfrak{g}, K)$ module.) For consistency of terminology, we often refer to the representation of π on V as "the representation V."

Even the correspondence $V \to C^\omega(V)_K$ is not one-one. For example, V might be the closure in a suitable norm of a G invariant space of functions on a homogeneous space of G. If the L^2 norm is used, one version of V results, while if an L^2 norm on the function and its first partial derivatives is used, another version of V results. Especially because of Theorem 0.6 below, it is customary to define away this problem. We say that V_1 and V_2 are **infinitesimally equivalent** if $C^\infty(V_1)_K$ and $C^\infty(V_2)_K$ are equivalent algebraically—i.e., if there is a $\mathbb{C}$ linear isomorphism of $C^\infty(V_1)_K$ onto $C^\infty(V_2)_K$ respecting the $U(\mathfrak{g})$ and K actions.

The reduction to algebra works best for representations that are irreducible or almost irreducible. Theorems 0.2 and 0.3 below prepare the setting. We say that V or its underlying $(\mathfrak{g}, K)$ module is **quasisimple** if the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ operates as scalars in the $(\mathfrak{g}, K)$ module. Theorem 0.2 should be regarded as a version of Schur's Lemma.

Theorem 0.2 (Segal, Mautner). If V is an irreducible unitary representation of G on a Hilbert space, then V is quasisimple.

If τ is an irreducible finite-dimensional representation of K, let V_τ be the sum of all K invariant subspaces of V_K for which the K action under π is equivalent with τ. We say that V is **admissible** if each V_τ is finite-dimensional. From the denseness of $C^\omega(V)_K$ in V given in Theorem 0.1, it follows that $C^\omega(V)_\tau$ is dense in V_τ. Consequently if V_τ is finite-dimensional, then $C^\omega(V)_\tau = C^\infty(V)_\tau = V_\tau$.

Theorem 0.3 (Harish-Chandra). If V is an irreducible quasisimple representation of G on a Banach space, then V is admissible.

Thus admissibility of a $(\mathfrak{g}, K)$ module is a reasonable way to make precise the idea of being almost irreducible. Theorem 0.1 has the following easy but important consequence.

Theorem 0.4 (Harish-Chandra). If V is an admissible representation of G on a Banach space, then the closed G invariant subspaces W of V stand in one-one correspondence with the $U(\mathfrak{g})$ invariant subspaces S of $V_K = C^\infty(V)_K$, the correspondence $W \leftrightarrow S$ being

$$S = W_K \qquad \text{and} \qquad W = \overline{S}.$$

For many purposes it is the unitary representations that are of primary interest. In the case of a unitary representation π on a Hilbert space V with Hermitian inner product $\langle \cdot, \cdot \rangle$, we see immediately that the underlying $(\mathfrak{g}, K)$ module $C^\infty(V)_K$ has the properties that

$$\text{(0.5)} \qquad \begin{aligned} \langle \pi(X)v_1, v_2 \rangle &= -\langle v_1, \pi(X)v_2 \rangle \quad &&\text{for } X \in \mathfrak{g}_0 \text{ and } v_1,\ v_2 \in V \\ \langle \pi(k)v_1, \pi(k)v_2 \rangle &= \langle v_1, v_2 \rangle &&\text{for } k \in K \text{ and } v_1,\ v_2 \in V. \end{aligned}$$

In the reverse direction, we say that a Hermitian form $\langle \cdot, \cdot \rangle$ on a $(\mathfrak{g}, K)$ module is **invariant** if (0.5) holds. The $(\mathfrak{g}, K)$ module is **infinitesimally unitary** if it admits a positive definite invariant Hermitian form.

Classifying irreducible unitary representations is the same as classifying irreducible admissible infinitesimally unitary $(\mathfrak{g}, K)$ modules, as a result of the following theorem. The theorem is due to Harish-Chandra for G linear. For general G, extra steps due independently to Lepowsky and Rader are needed for the proof.

Theorem 0.6.

(a) Any irreducible admissible infinitesimally unitary $(\mathfrak{g}, K)$ module is the underlying $(\mathfrak{g}, K)$ module of an irreducible unitary representation of G on a Hilbert space.

(b) Two irreducible unitary representations of G on Hilbert spaces are unitarily equivalent if and only if they are infinitesimally equivalent.

2. Early Constructions of Representations

One of the fundamental problems in the representation theory of semisimple groups is to classify and categorize the irreducible unitary representations. Bargmann classified the irreducible unitary representations of

$$G = SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\}$$

by classifying the candidates for underlying $(\mathfrak{g}, K)$ modules and then exhibiting unitary representations corresponding to each. About the time of Bargmann's work, Gelfand and Naimark classified the irreducible unitary representations of $SL(2, \mathbb{C})$ by using global methods. The representations for these two groups were later taken by other people as models for constructions in other semisimple groups.

Let us describe two of the series of representations obtained by Bargmann. The first of these, now known as the **principal series**, consists of one representation of $G = SL(2, \mathbb{R})$ for each parameter $(\pm, iv)$, where $\pm$ is a sign and iv is a purely imaginary complex number. The space in each case is $L^2(\mathbb{R})$, and the action for the representation with parameter $(\pm, iv)$ is

$$\left(\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(x) = \begin{cases} |-bx+d|^{-1-iv} f(\frac{ax-c}{-bx+d}) & \text{if } + \\ \operatorname{sgn}(-bx+d)|-bx+d|^{-1-iv} f(\frac{ax-c}{-bx+d}) & \text{if } -. \end{cases}$$

For the second series, now known as the **discrete series**, it is more convenient to work with the isomorphic group

$$G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \,\middle|\, |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

This series consists of one representation of G for each parameter $(\pm, n)$, where $\pm$ is a sign and n is an integer ≥ 2. For the series with the sign $-$, the Hilbert space is the set of analytic f in the unit disc for which

$$\|f\|^2 = \int_{|z|<1} |f(z)|^2 (1-|z|^2)^{n-2}\, dx\, dy < \infty,$$

and the action is

$$\left(\pi \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} f\right)(z) = (-\bar\beta z + \alpha)^{-n} f\left(\frac{\bar\alpha z - \beta}{-\bar\beta z + \alpha}\right).$$

The series with the sign $+$ is obtained by taking the complex conjugate of $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ before applying π as above.

Meanwhile Mackey was developing a theory of induced representations for locally compact groups, and he was apparently the first to realize that the principal-series representations of $SL(2, \mathbb{R})$ were of this form. (See the Notes.) More particularly the principal series were a case of what we shall call "parabolic induction." An account of the general case appears in Chapter XI below. In the special case of $SL(2, \mathbb{R})$, define subgroups of $G = SL(2, \mathbb{R})$ by

$$K = \{k_\theta\} = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right\},$$

$$M = \left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \quad A = \left\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right\}, \quad N = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\}.$$

To the parameter $(\pm, iv)$, we associate a one-dimensional character of M and the differential of a one-dimensional character of A by

$$\sigma\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} = \begin{cases} \varepsilon & \text{if } \pm \text{ is } - \\ 1 & \text{if } \pm \text{ is } + \end{cases}$$

and

$$\nu\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = ivt.$$

Then $man \mapsto e^{\nu \log a}\sigma(m)$ is a representation of the upper triangular subgroup MAN, and we shall define the corresponding classical induced representation $\tilde\pi$ of G. The definition of $\tilde\pi$ involves a shift in parameter by

$$\rho\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = t$$

in order to get the representation $\tilde{\pi}$ to be unitary. A dense subspace of the Hilbert space is

$$\{F : G \to \mathbb{C} \text{ of class } C^\infty \mid F(xman) = e^{-(\nu+\rho)\log a}\sigma(m)^{-1}F(x)\}$$

with

$$\|F\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |F(k_\theta)|^2\,d\theta,$$

and the action is simply

$$(\tilde{\pi}(g)F)(x) = F(g^{-1}x).$$

The actual Hilbert space is the completion of the above dense subspace, with action by the continuous extension of each $\tilde{\pi}(g)$. The correspondence $F \mapsto f$ with the more classical realization is given by $f(y) = F\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$, except that a constant factor needs to be introduced if the correspondence is to be unitary.

Generalizing the principal series to semisimple groups is then just a matter of a little structure theory. The upper triangular subgroup gets replaced by a parabolic subgroup of G, this parabolic subgroup decomposes suitably as MAN, σ gets replaced by a suitable kind of irreducible unitary representation of M, and ν gets replaced by an imaginary-valued linear functional on the Lie algebra of A. The result is **parabolic induction**. When ρ is correctly generalized, parabolic induction carries unitary representations to unitary representations.

It was Harish-Chandra who found how to generalize the discrete series. While the generalization of the principal series used real analysis, the generalization of the discrete series required complex analysis. For the group $G = SU(1,1)$, the analytic group of matrices with Lie algebra $\mathfrak{g}$ is $G_{\mathbb{C}} = SL(2,\mathbb{C})$. Within $G_{\mathbb{C}}$, let

$$B = \left\{\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}\right\}.$$

Then we readily check that

(a) every element of the subset $GB \subseteq G_{\mathbb{C}}$ has a unique decomposition as a product

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma^{-1} & 0 \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} \quad \text{with } \zeta \in \mathbb{C},\ \gamma \in \mathbb{C}^\times,\ |z| < 1, \tag{0.7}$$

and every matrix (0.7) is in GB.

(b) GB is an open subset of $SL(2,\mathbb{C})$, and its product complex structure obtained from (0.7) is the same as what it inherits from $G_{\mathbb{C}}$. In particular, left translation by any member of G is a holomorphic automorphism of GB.

To construct the generalizable version of the discrete series with parameter $(-, n)$, let ξ_n be the one-dimensional holomorphic representation of B given by

$$\xi_n \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = a^{-n}.$$

The Hilbert space is taken as

(0.8)
$$\left\{ F : GB \to \mathbb{C} \;\middle|\; \begin{array}{ll} \text{(i)} & F \text{ is holomorphic} \\ \text{(ii)} & F(xb) = \xi_n(b)^{-1}F(x) \text{ for } x \in GB,\ b \in B \\ \text{(iii)} & \|F\|^2 = \int_G |F(x)|^2\,dx < \infty \end{array} \right\},$$

and the action is

$$(\tilde{\pi}(g)F)(x) = F(g^{-1}x) \qquad \text{for } F \text{ as in (0.8)},\ g \in G,\ x \in GB.$$

Except for a constant depending on the normalization of dx, the correspondence $F \mapsto f$ with the more classical realization is given by $f(z) = F(z, 1, 0)$ relative to the coordinates (0.7), and the inverse is $f \mapsto F$ with $F(z, \gamma, \zeta) = \gamma^{-n} f(z)$.

The generalization of this construction to other semisimple groups involves some special assumption on the group, and the resulting representations (when nonzero) are called the "holomorphic discrete series." If G is noncompact simple, the special assumption is that G/K is a complex manifold on which G operates holomorphically, and then G/K arises from the generalization of (0.7) as the z's allowed in the matrices $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$. But Harish-Chandra phrased the condition in terms of roots. He assumed that a maximal torus of K is maximal abelian in G, and he hypothesized that in some ordering on the roots "every noncompact positive root is totally positive." With the terminology of roots stripped away, this condition says that the centralizer in G of the identity component of the center of K is K itself.

At any rate the condition is satisfied if G is compact connected, and Harish-Chandra's construction therefore gives global realizations of the irreducible representations of compact connected groups. Other authors came upon the same realizations of representations of compact connected groups independently at about the same time, starting from the point of view of algebraic geometry, and the result has come to be known as the Borel-Weil Theorem. Let us give a precise statement. The irreducible representations of a compact connected G are given by the Theorem of the Highest Weight, and we shall describe a global realization in terms of the highest weight as parameter. Regard G as a matrix group, let $G_{\mathbb{C}}$ be its complexification, introduce a maximal torus T of G, and fix a system of positive roots. Let B be the analytic subgroup of $G_{\mathbb{C}}$ whose Lie algebra contains the complexified Lie algebra $\mathfrak{t}$ of T, as well as the root spaces for all the negative roots. It turns out that GB is open and closed in $G_{\mathbb{C}}$ and hence $G_{\mathbb{C}} = GB$.

Theorem 0.9 (Borel-Weil Theorem). For the compact connected Lie group G, if $\lambda \in \mathfrak{t}^*$ is dominant and analytically integral and if ξ_λ denotes the corresponding holomorphic one-dimensional representation of B, then a realization of an irreducible representation of G with highest weight λ is in the space

(0.10)
$$\left\{ F : GB \to \mathbb{C} \;\middle|\; \begin{array}{ll} \text{(i)} & F \text{ is holomorphic} \\ \text{(ii)} & F(xb) = \xi_\lambda(b)^{-1} F(x) \text{ for } x \in GB,\ b \in B \\ \text{(iii)} & \|F\|^2 = \int_G |F(x)|^2\, dx < \infty \end{array} \right\}$$

with G acting by

$$(\tilde{\pi}(g)F)(x) = F(g^{-1}x) \qquad \text{for } F \text{ as in (0.10)},\ g \in G,\ x \in GB.$$

Condition (iii) is automatic in the presence of (i), and we can drop it. Also we can replace GB by $G_{\mathbb{C}}$, but we prefer to emphasize the parallel with the construction for G noncompact by leaving GB in place.

From a geometric point of view the setting underlying the Borel-Weil Theorem is a bundle that we can view two ways

$$\begin{array}{ccc} GB & \longrightarrow & G_{\mathbb{C}} \\ \downarrow & & \downarrow \\ G/T & \longrightarrow & G_{\mathbb{C}}/B \end{array}$$

The left column is the one of interest for representations, and the map is the quotient by B. The horizontal maps are inclusions with image open since $G \cap B = T$, and they are onto since G is compact. The map on the right is the quotient map by the closed complex subgroup B. At the bottom right the quotient $G_{\mathbb{C}}/B$ is a complex manifold, and G/T therefore acquires an invariant complex structure. In a way that will be described in the next section, the functions F of (0.10) may be identified with the holomorphic sections of the holomorphic line bundle over G/T

associated to the character ξ_λ, and G acts on the space of sections in the natural way. In short, the irreducible representation with highest weight λ is realized as the space of global holomorphic sections of a certain holomorphic line bundle.

Suppose now that λ is analytically integral but no longer dominant. The space (0.10) still makes sense, but it is now zero, i.e., the holomorphic line bundle has no nontrivial sections. In order to find an interesting representation, we need an additional idea.

In complex geometry an operator $\bar\partial$ allows the introduction of a cohomology theory in such a way that the 0^{th}-degree cohomology is just the space of global holomorphic sections. The operator $\bar\partial$ has a formula like that of the deRham d, except that $\dfrac{\partial}{\partial x_j}$ and dx_j get replaced by $\dfrac{\partial}{\partial \bar z_j}$ and $d\bar z_j$. We shall describe $\bar\partial$ more precisely in the next section.

In any event, in the holomorphic line bundle associated to the character ξ_λ, it is possible to speak of smooth $(0,k)$ cochain sections, and the image of one level of $\bar\partial$ is contained in the kernel of the next level. Let the representation space of the one-dimensional representation ξ_λ be denoted $\mathbb{C}_\lambda$, and let the spaces of cocycles and coboundaries be called

$$Z^{0,k}(G/T,\mathbb{C}_\lambda) \qquad \text{and} \qquad B^{0,k}(G/T,\mathbb{C}_\lambda),$$

respectively. The group G acts on these, and the quotient

$$H^{0,k}(G/T,\mathbb{C}_\lambda) = Z^{0,k}(G/T,\mathbb{C}_\lambda)/B^{0,k}(G/T,\mathbb{C}_\lambda)$$

is called the $(0,k)^{\text{th}}$ space of Dolbeault cohomology sections. From the point of view of representation theory, it is desirable to have a topology on these spaces such that the topology on $H^{0,k}(G/T,\mathbb{C}_\lambda)$ is obtained as the quotient topology from the other two. The Hilbert-space topology on $Z^{0,k}(G/T,\mathbb{C}_\lambda)$ from square integrability on G is not convenient, because use of the completion makes it necessary to address the meaning of $\bar\partial$ on nonsmooth cochain sections. But we shall be able to give $Z^{0,k}(G/T,\mathbb{C}_\lambda)$ a satisfactory C^∞ type topology below. Since $H^{0,k}(G/T,\mathbb{C}_\lambda)$ will be Hausdorff if and only if $B^{0,k}(G/T,\mathbb{C}_\lambda)$ is a closed subspace, it is important to know whether $B^{0,k}(G/T,\mathbb{C}_\lambda)$ is closed in $Z^{0,k}(G/T,\mathbb{C}_\lambda)$. For G compact it is indeed closed, but it is not a trivial matter to prove this fact. The Bott-Borel-Weil Theorem identifies the space $H^{0,k}(G/T,\mathbb{C}_\lambda)$. The notation is

(0.11)
$$\begin{aligned}
&\Delta = \{\text{roots of } (\mathfrak{g},\mathfrak{t})\}\\
&\Delta^+ = \text{a positive system for } \Delta\\
&\delta = \tfrac12 \textstyle\sum_{\alpha\in\Delta^+}\alpha\\
&W = \text{Weyl group of } \Delta\\
&B = \text{Borel subgroup built from } \textit{negative} \text{ roots}\\
&G/T\text{'s complex structure from } G_{\mathbb{C}}/B.
\end{aligned}$$

Theorem 0.12 (Bott-Borel-Weil Theorem, first form). With G compact and with notation as in (0.11), and let $\lambda \in \mathfrak{t}^*$ be integral.

(a) If $\langle \lambda + \delta, \alpha \rangle = 0$ for some $\alpha \in \Delta$, then $H^{0,k}(G/T, \mathbb{C}_\lambda) = 0$ for all k.

(b) If $\langle \lambda + \delta, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$, let

$$q = \#\{\alpha \in \Delta^+ \mid \langle \lambda + \delta, \alpha \rangle < 0\}. \tag{0.13}$$

Choose $w \in W$ with $w(\lambda + \delta)$ dominant, and put $\mu = w(\lambda + \delta) - \delta$. Then

$$H^{0,k}(G/T, \mathbb{C}_\lambda) = \begin{cases} 0 & \text{if } k \neq q \\ F^\mu & \text{if } k = q, \end{cases}$$

where F^μ is a finite-dimensional irreducible representation of G with highest weight μ.

Before taking up the detailed discussion of representations in spaces of Dolbeault cohomology sections, let us generalize our definition of representation suitably. Let V be a locally convex, complete, complex linear topological (Hausdorff) space. A (continuous) **representation** of the Lie group G on V is a homomorphism $\pi : G \to \operatorname{Aut} V$ such that the map $G \times V \to V$ is continuous. With no change in the formalism, we can define the subspace $C^\infty(V)$ of C^∞ **vectors**. The assumption that V is locally convex and complete allows us to define the integral of a continuous function from a compact Hausdorff space X into V, with respect to a Borel measure on X. Taking X to be a compact neighborhood of 1 in G, we can apply Gårding's argument given above to see that $C^\infty(V)$ is dense in V.

For G semisimple with maximal compact subgroup K, we can again speak of the subspace V_K of K finite vectors. If τ is an irreducible finite-dimensional representation of K and if Θ_{τ^*} and d_τ are the character and degree of the contragredient of τ, define

$$\pi(\chi_\tau)v = \int_K d_\tau \Theta_{\tau^*}(k)\pi(k)v \, dk \qquad \text{for } v \in V.$$

Then $\pi(\chi_\tau)$ is a continuous projection whose image is V_τ and whose kernel contains all $V_{\tau'}$ for τ' inequivalent with τ. With this definition in place, we can argue that V_K is dense in V and that $C^\infty(V)_\tau$ is dense in V_τ.

A representation π on V as above is said to be **smooth** if $C^\infty(V) = V$. When a representation is given to us on a Banach space, the subspace of C^∞ vectors becomes a smooth representation if $C^\infty(V)$ is retopologized using the family of seminorms $\|\cdot\|_u$ parametrized by $u \in U(\mathfrak{g})$ and defined by $\|v\|_u = \|\pi(u)v\|$. Any smooth representation becomes a $U(\mathfrak{g})$ module under the definition $u \cdot v = \pi(u)v$, and its subspace of K finite vectors is a $(\mathfrak{g}, K)$ module called the **underlying** $(\mathfrak{g}, K)$ **module** of V.

3. Sections of Homogeneous Vector Bundles

This section describes a representation-theoretic construction by complex analysis that generalizes what happens for the holomorphic discrete series and the Bott-Borel-Weil Theorem. The expectation is that many of the resulting representations will be irreducible unitary and that we will therefore have a complex-analysis construction to complement the real-analysis construction given by parabolic induction. It is assumed that the reader is acquainted with some elementary structure theory of semisimple groups; discussion of this topic may be found in Chapter IV below. We shall make use of vector bundles in the construction. Although a full analytic theory requires understanding vector bundles with infinite-dimensional fiber, we shall restrict to the finite-dimensional case.

Throughout this section we work with the following setting, sometimes limiting ourselves to special cases: G is a connected linear reductive Lie group with complexification $G_{\mathbb{C}}$, K is a fixed maximal compact subgroup, T is a compact connected abelian subgroup of K (hence a torus), and $L = Z_G(T)$ is the centralizer of T in G. From Lemma 5.10 below, it is known that L is connected. Therefore the complexification $L_{\mathbb{C}}$ is meaningful as a subgroup of $G_{\mathbb{C}}$, namely the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra the complex subalgebra generated by the Lie algebra of L. Let Q be a parabolic subgroup of $G_{\mathbb{C}}$ with Levi factor $L_{\mathbb{C}}$.

We denote Lie algebras of Lie groups $A,\ B$, etc., by $\mathfrak{a}_0,\ \mathfrak{b}_0$, etc., and we denote their complexifications by $\mathfrak{a},\ \mathfrak{b}$, etc. The complex Lie algebras of complex Lie groups $G_{\mathbb{C}},\ L_{\mathbb{C}},\ Q$ are denoted $\mathfrak{g},\ \mathfrak{l},\ \mathfrak{q}$. We use an overbar to denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

We can decompose the Lie algebra $\mathfrak{q}$ of Q as a vector-space direct sum $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{u}$ is the nilradical. Then $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are both nilpotent complex Lie algebras, and we have $[\mathfrak{l}, \mathfrak{u}] \subseteq \mathfrak{u}$ and $[\mathfrak{l}, \bar{\mathfrak{u}}] \subseteq \bar{\mathfrak{u}}$.

We assume that $\mathfrak{q}$ is a θ **stable parabolic**; this condition means that

$$\mathfrak{g}_0 \cap \mathfrak{q} = \mathfrak{l}_0. \tag{0.14a}$$

It is equivalent to assume a vector-space direct-sum decomposition

$$\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u}. \tag{0.14b}$$

Under the condition (0.14), the natural mapping $G/L \to G_{\mathbb{C}}/Q$ is an inclusion, and the image is an open set. Thus the choice of Q has made G/L into a complex manifold with G operating holomorphically.

A noncompact example to keep in mind is the group $G = U(m, n)$ of complex matrices that preserve an indefinite Hermitian form. Here $G_{\mathbb{C}} = GL(m+n, \mathbb{C})$. If we take T to be any closed connected subgroup of the diagonal of the form

$$T = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_2}, \cdots, e^{i\theta_r}, \dots, e^{i\theta_r}),$$

then L will be a block-diagonal subgroup within G with r blocks, and L will necessarily be connected. We can choose $\mathfrak{u}$ to be the complex Lie algebra of corresponding block-upper-triangular matrices and $\bar{\mathfrak{u}}$ to consist of the corresponding block-lower-triangular matrices.

We take as known that

$$p : G \to G/L \tag{0.15}$$

is a C^∞ principal fiber bundle with structure group L. Let V be a finite-dimensional real or complex vector space, let $GL(V)$ be its general linear group, and let $\rho : L \to GL(V)$ be a C^∞ homomorphism. The **associated vector bundle**

$$p_V : G \times_L V \to G/L \tag{0.16a}$$

is a vector bundle with structure group $GL(V)$ whose bundle space is given by

$$G \times_L V = \{(g, v)/\sim\} \qquad \text{with} \qquad (gl, v) \sim (g, \rho(l)v) \tag{0.16b}$$

for $g \in G$, $l \in L$, and $v \in V$. Let $[(g, v)]$ denote the class of (g, v). We omit a description of the bundle structure.

The space of C^∞ sections of (0.16) is denoted $\mathcal{E}(G \times_L V)$. The group G acts on $G \times_L V$ by left translation: $g_0[(g, v)] = [(g_0 g, v)]$ in the notation of (0.16b). This action induces a well-defined action of G on $\mathcal{E}(G \times_L V)$ by $(g_0\gamma)(gL) = g_0(\gamma(g_0^{-1}gL))$ for $\gamma \in \mathcal{E}(G \times_L V)$. When V is complex, this construction yields a representation of G (understood to be on a complex vector space). This representation is continuous in the sense that $(g_0, \gamma) \mapsto g_0\gamma$ is continuous from $G \times \mathcal{E}(G \times_L V)$ to $\mathcal{E}(G \times_L V)$ if $\mathcal{E}(G \times_L V)$ is given its usual C^∞ topology. It is a smooth representation in the sense of §2.

Similarly

$$p : G_{\mathbb{C}} \to G_{\mathbb{C}}/Q \tag{0.17}$$

is a holomorphic principal fiber bundle with structure group Q. In the above situation if V is complex and if ρ extends to a holomorphic homomorphism $\rho : Q \to GL(V)$, then we can construct an associated vector bundle

(0.18a) $$p_V : G_{\mathbb{C}} \times_{\mathbb{C}} V \to G_{\mathbb{C}}/Q$$

with bundle space given by

(0.18b) $$G_{\mathbb{C}} \times_Q V = \{(g_{\mathbb{C}}, v)/\sim\} \qquad \text{with} \qquad (g_{\mathbb{C}}q, v) \sim (g_{\mathbb{C}}, \rho(q)v).$$

The bundle (0.18) is a holomorphic vector bundle.

The inclusion $G/L \hookrightarrow G_{\mathbb{C}}/Q$ induces via pullback from (0.18a) a bundle map

(0.19) $$G \times_L V \hookrightarrow G_{\mathbb{C}} \times_Q V.$$

In terms of (0.16b) and (0.18b), this map is given simply by $(g, v) \mapsto (g, v)$. The result is that the C^∞ complex vector bundle $G \times_L V$ acquires the structure of a holomorphic vector bundle. We can regard the space of holomorphic sections $\mathcal{O}(G \times_L V)$ of $G \times_L V$ as a vector subspace of $\mathcal{E}(G \times_L V)$. (Actually less is needed about ρ than extendibility to Q in order to get the homomorphic structure on $G \times_L V$. See the Notes for details.)

To any section γ of $G \times_L V$ we can associate a function $\varphi_\gamma : G \to V$ by the definition

(0.20a) $$\gamma(gL) = [(g, \varphi_\gamma(g))] \in G \times_L V.$$

Under this correspondence, C^∞ sections γ go to C^∞ functions φ_γ, and we obtain an isomorphism

(0.20b)
$$\mathcal{E}(G \times_L V) \cong \left\{ \varphi : G \to V \;\middle|\; \begin{array}{l} \varphi \text{ of class } C^\infty, \\ \varphi(gl) = \rho(l)^{-1}\varphi(g) \text{ for } l \in L,\ g \in G \end{array} \right\}.$$

The group G acts on the right member of (0.20b) by the left regular action, and the isomorphism respects the actions by G. The usual C^∞ topology on $\mathcal{E}(G \times_L V)$ corresponds to the C^∞ topology on the space of φ's. It is under this correspondence that we can identify the functions F in (0.8) and (0.10) with sections of holomorphic line bundles.

The correspondence $\gamma \leftrightarrow \varphi_\gamma$ works locally as well, with sections over an open set $U \subseteq G/L$ corresponding to functions φ on the open subset $p^{-1}(U)$ of G transforming as in (0.20b). Again γ of class C^∞ corresponds to φ_γ of class C^∞. Let $\mathcal{E}(U)$ be the space of C^∞ sections over U.

In the special case that $G \times_L V$ admits the structure of a holomorphic vector bundle because of (0.19) and (0.18), we can speak of the space of holomorphic sections $\mathcal{O}(U)$ over an open set $U \subseteq G/L$. The proposition below tells how to use φ_γ to decide whether γ is holomorphic.

Proposition 0.21. Suppose that ρ extends to a holomorphic homomorphism $\rho : Q \to GL(V)$ and thereby makes $G \times_L V$ into a holomorphic vector bundle. Let $U \subseteq G/L$ be open, let γ be in $\mathcal{E}(U)$, and let φ_γ be the corresponding function from $p^{-1}(U)$ to V given by (0.20). Then γ is holomorphic if and only if

$$(Z\varphi_\gamma)(g) = -\rho(Z)(\varphi_\gamma(g)) \tag{0.22a}$$

for all $g \in p^{-1}(U)$ and $Z \in \mathfrak{q}$, with Z acting on φ_γ as a complex left-invariant vector field.

In typical applications to representation theory, ρ in the proposition is given on L and extends holomorphically to $L_{\mathbb{C}}$. The extension to Q is taken to be trivial on the unipotent radical of Q. Equation (0.22a) holds for $Z \in \mathfrak{l}_0$ for any C^∞ section, and it extends to $Z \in \mathfrak{l}$ by complex linearity. Thus (0.22a) may be replaced in this situation by the condition

$$Z\varphi_\gamma = 0 \qquad \text{for all } Z \in \mathfrak{u}. \tag{0.22b}$$

The special case $\rho = 1$ shows how to recognize holomorphic functions on open subsets of G/L.

Let M be a complex manifold, and let p be in M. We denote by $T_p(M)$ the tangent space of M (considered as a C^∞ manifold) at p, consisting of derivations of the algebra of smooth germs at p, and we let $T(M)$ be the tangent bundle. Also we denote by $T_{\mathbb{C},p}(M)$ the complex vector space of derivations of the algebra of holomorphic germs at p, and we let $T_{\mathbb{C}}(M)$ be the corresponding bundle. There is a canonical $\mathbb{R}$ isomorphism

$$T_p(M) \xrightarrow{\sim} T_{\mathbb{C},p}(M) \tag{0.23a}$$

given by

$$\xi \mapsto \zeta, \qquad \text{where } \zeta(u + iv) = \xi(u) + i\xi(v). \tag{0.23b}$$

Let J_p be the member of $GL(T_p(M))$ that corresponds under (0.23) to multiplication by i in $T_{\mathbb{C},p}(M)$. Then $J = \{J_p\}$ is a bundle map from $T(M)$ to itself whose square is -1.

The following proposition allows us to relate these considerations to associated vector bundles.

Proposition 0.24. There are canonical bundle isomorphisms

$$T(G/L) \cong G \times_L (\mathfrak{g}_0/\mathfrak{l}_0) \tag{0.25a}$$

and

$$T_{\mathbb{C}}(G_{\mathbb{C}}/Q) \cong G_{\mathbb{C}} \times_Q (\mathfrak{g}/\mathfrak{q}) \tag{0.25b}$$

with L and Q acting on $\mathfrak{g}_0/\mathfrak{l}_0$ and $\mathfrak{g}/\mathfrak{q}$, respectively, by Ad.

The inclusion $G/L \subseteq G_{\mathbb{C}}/Q$ allows us to regard

$$T_{\mathbb{C}}(G/L) \cong GQ \times_Q (\mathfrak{g}/\mathfrak{q}). \tag{0.26}$$

At any point $p = gL$ of G/L, the left sides of (0.25a) and (0.26), namely $T(G/L)$ and $T_{\mathbb{C}}(G/L)$, are $\mathbb{R}$ isomorphic via (0.23). It is easy to check that the corresponding isomorphism of the right sides of (0.25a) and (0.26) at p is given by

$$(g, X + \mathfrak{l}_0) \mapsto (g, X + \mathfrak{q}) \qquad \text{for } g \in G,\ X \in \mathfrak{g}_0.$$

This result allows us to compute the effect of J.

Complexifying (0.25a), we have

$$T(G/L)_{\mathbb{C}} \cong G \times_L (\mathfrak{g}_0/\mathfrak{l}_0)_{\mathbb{C}},$$

and J acts in the fiber at each point. We let $T(G/L)^{1,0}$ and $T(G/L)^{0,1}$ be the subbundles of $T(G/L)_{\mathbb{C}}$ corresponding to the respective eigenvalues i and $-i$ of J, so that

$$T(G/L)_{\mathbb{C}} \cong T(G/L)^{1,0} \oplus T(G/L)^{0,1}. \tag{0.27a}$$

We have

$$(\mathfrak{g}_0/\mathfrak{l}_0)_{\mathbb{C}} \cong \mathfrak{g}/\mathfrak{l} \cong \bar{\mathfrak{u}} \oplus \mathfrak{u} \tag{0.27b}$$

as L modules, and a little calculation shows that (0.27b) gives the decomposition of the fibers under J corresponding to (0.27a). In other words

$$\begin{aligned} T(G/L)^{1,0} &\cong G \times_L \bar{\mathfrak{u}} \\ T(G/L)^{0,1} &\cong G \times_L \mathfrak{u}. \end{aligned} \tag{0.27c}$$

Taking duals in (0.27a) and forming alternating tensors, we have

$$\wedge^{p,q} T^*(G/L)_{\mathbb{C}} \cong G \times_L ((\wedge^p \bar{\mathfrak{u}})^* \otimes (\wedge^q \mathfrak{u})^*). \tag{0.28}$$

Via (0.28), members of $\mathcal{E}(\wedge^{p,q} T^*(G/L)_{\mathbb{C}})$ correspond to functions from G to $(\wedge^p \bar{\mathfrak{u}})^* \otimes (\wedge^q \mathfrak{u})^*$ transforming on the right under L by $\mathrm{Ad}^* \otimes \mathrm{Ad}^*$.

The scalar $\bar{\partial}$ operator for a complex manifold M is an operator

$$\bar{\partial} : \mathcal{E}(\wedge^{p,q} T^*(M)_{\mathbb{C}}) \to \mathcal{E}(\wedge^{p,q+1} T^*(M)_{\mathbb{C}}),$$

and it has $\bar\partial^2 = 0$. For the case that $M = G/L$, we can interpret $\bar\partial$ in terms of (0.28).

We can construct also a vector-valued version of $\bar\partial$. Namely let $G \times_L V$ be a holomorphic vector bundle as above. We introduce $\bar\partial_V = \bar\partial \otimes 1$ as an operator

$$\bar\partial_V : \mathcal{E}(\wedge^{p,q} T^*(G/L)_{\mathbb{C}} \otimes (G \times_L V)) \to \mathcal{E}(\wedge^{p,q+1} T^*(G/L)_{\mathbb{C}} \otimes (G \times_L V));$$

$\bar\partial_V$ is well defined because the transition functions for $G \times_L V$ are holomorphic. Also $\bar\partial_V^2 = 0$. Using (0.28) and dropping the subscript "V" on $\bar\partial_V$, we can interpret $\bar\partial_V$ as an operator

$$\bar\partial : \mathcal{E}(G \times_L ((\wedge^p \bar{\mathfrak{u}})^* \otimes (\wedge^q \mathfrak{u})^* \otimes V)) \to \mathcal{E}(G \times_L ((\wedge^p \bar{\mathfrak{u}})^* \otimes (\wedge^{q+1} \mathfrak{u})^* \otimes V)).$$

In representation theory one works with the case $p = 0$. We define

$$C^{0,q}(G/L, V) = \mathcal{E}(G \times_L ((\wedge^q \mathfrak{u})^* \otimes V)).$$

As always, this is the representation space for a continuous representation of G. The operator $\bar\partial$ is continuous and the kernel is closed. H.-W. Wong has shown (under the standing hypothesis of finite-dimensional V) that the image of $\bar\partial$ is closed and therefore that the quotient is Hausdorff. Thus we can define the **Dolbeault cohomology space** $H^{0,q}(G/L, V)$ as

$$H^{0,q}(G/L, V) = \ker(\bar\partial|_{C^{0,q}(G/L,V)})/\mathrm{image}(\bar\partial|_{C^{0,q-1}(G/L,V)}). \tag{0.29}$$

Since $\bar\partial$ commutes with G, the topological vector space $H^{0,q}(G/L, V)$ carries a continuous representation of G.

The Bott-Borel-Weil Theorem identifies the spaces $H^{0,q}(G/L, V)$ of (0.29) in the case that G is compact. In this situation it has long been known that $\bar\partial$ has closed image. If one introduces a Hermitian inner product on V, then the formal adjoint $\bar\partial^*$ of $\bar\partial$ is meaningful, and it has long been known also that (0.29) can be computed alternatively as the representation on $\ker\bar\partial \cap \ker\bar\partial^*$ in $C^{0,q}(G/L, V)$. Members of $\ker\bar\partial \cap \ker\bar\partial^*$ are called **strongly harmonic**; this alternate approach shows that each cohomology class has exactly one strongly harmonic representative.

We have already stated the Bott-Borel-Weil Theorem in the special case that $L = T$ and $Q = B$. For general G/L with G compact and L the centralizer of a torus, the notation is

(0.30)
$$\begin{aligned}
&G = \text{compact connected Lie group}\\
&T = \text{a torus in } G\\
&L = Z_G(T)\\
&T \text{ extended to a maximal torus } \tilde{T} \text{ in } L\\
&\Delta = \{\text{roots of } (\mathfrak{g}, \tilde{\mathfrak{t}})\}\\
&\Delta(\mathfrak{l}) = \{\text{roots of } (\mathfrak{l}, \tilde{\mathfrak{t}})\} \subseteq \Delta\\
&\Delta^+ \text{ chosen with } \Delta(\mathfrak{l}) \text{ generated by simple roots}\\
&\delta = \tfrac{1}{2} \textstyle\sum_{\alpha \in \Delta^+} \alpha\\
&W = \text{Weyl group}\\
&Q = \text{built from } \mathfrak{l} \text{ and } \textit{negative} \text{ roots}\\
&G/L\text{'s complex structure from } G_{\mathbb{C}}/Q.
\end{aligned}$$

Theorem 0.31 (Bott-Borel-Weil Theorem, second form). With G compact and with notation as in (0.30), let V^λ be irreducible for L with highest weight λ.

(a) If $\langle \lambda + \delta, \alpha \rangle = 0$ for some $\alpha \in \Delta$, then $H^{0,j}(G/L, V^\lambda) = 0$ for all j.

(b) If $\langle \lambda + \delta, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$, define q as in (0.13), choose $w \in W$ so that $w(\lambda + \delta)$ is dominant, and put $\mu = w(\lambda + \delta) - \delta$. Then

$$H^{0,j}(G/L, V^\lambda) = \begin{cases} 0 & \text{if } j \neq q \\ F^\mu & \text{if } j = q, \end{cases}$$

where F^μ is a finite-dimensional irreducible representation of G with highest weight μ.

Historically the next cases of our construction to be considered were those for discrete-series representations. For a unimodular group G, an irreducible unitary representation π is in the **discrete series** if it is a direct summand of the right regular representation on $L^2(G)$, or equivalently if some (or equivalently every) nonzero matrix coefficient $(\pi(g)v_1, v_2)$ is in $L^2(G)$. Holomorphic discrete series for $SU(1,1)$ as in (0.7) and (0.8) provide examples.

Let G be linear connected semisimple, and let K be a maximal compact subgroup. For G compact (so that $K = G$), every irreducible unitary representation is in the discrete series. For G noncompact, the discrete-series representations were parametrized by Harish-Chandra. We shall not recite the parametrization now, but it has features in common with the Theorem of the Highest Weight and the Weyl character formula. At this time we need to know only that discrete-series representations exist for G if and only if a maximal torus T of K is maximal abelian in G.

Langlands conjectured that all of Harish-Chandra's discrete series could be realized globally in a fashion similar to that in the Bott-Borel-Weil Theorem (with base space G/T). In making this conjecture, Langlands imposed square integrability on his allowable cocycles and coboundaries. The virtue of this choice is that it makes the conjecture correct (as was later shown by Schmid); the difficulty is that parallel square-integrability restrictions are not available in the general setting of (0.29) when L is noncompact.

The problem with allowing arbitrary cocycles and coboundaries can already be seen in $SU(1,1)$. Since the unit disc is a Stein manifold, we can get nonzero cohomology only in degree 0 (by H. Cartan's Theorem B). Thus if we fix the one-dimensional holomorphic representation ξ_n of B, the interest is the space of functions $F : GB \to \mathbb{C}$ satisfying (i) and (ii) in (0.8). A feature of the theory of holomorphic discrete series is that all nonzero K finite F's satisfying (i) and (ii) also satisfy (iii), or else none do. When $n > 1$, (iii) holds and we get a unitary representation. But when $n \leq 1$, (iii) fails. For example, when $n = -1$, the space of functions F has a two-dimensional invariant subspace equivalent with the standard representation of $SU(1,1)$, which is not unitary.

It would be nice to have a setting where the L^2 cohomology and the Dolbeault cohomology are compatible, and Schmid discovered such a setting. His idea was to adapt Δ^+ (and hence the complex structure) to the parameter, making the parameter dominant. Then the degree of interest for cohomology is $S = \dim_{\mathbb{C}}(K/T) = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$. Under some hypotheses Schmid proved that the natural map from L^2 cohomology in degree S into Dolbeault cohomology is one-one.

If we rephrase the Bott-Borel-Weil Theorem with this idea in place, the notation is as follows: We let G, T, and Δ be as in (0.11). Let $\lambda_0 \in \mathfrak{t}^*$ be a given nonsingular parameter (λ_0 corresponds to $\lambda + \delta$ in Theorem 0.31), and suppose that $\lambda_0 - \delta_0$ is analytically integral for the half sum δ_0 of positive roots in some (or equivalently each) positive system. Define

(0.32)

$$\Delta^+ = \{\alpha \in \Delta \mid \langle \lambda_0, \alpha \rangle > 0\}$$
$$\delta = \tfrac{1}{2} \textstyle\sum_{\alpha \in \Delta^+} \alpha$$
$$\lambda = \lambda_0 - \delta$$

V^λ = irreducible finite-dimensional representation of L with highest weight λ

Q built from $\mathfrak{l}$ and Δ^+ instead of $-\Delta^+$

G/L's complex structure from $G_{\mathbb{C}}/Q$.

Theorem 0.33 (Bott-Borel-Weil Theorem, third form). Let G be compact connected, with notation as in (0.32). Then

$$H^{0,j}(G/L,\ V^\lambda \otimes_{\mathbb{C}} \textstyle\bigwedge^{\text{top}}\mathfrak{u}) = \begin{cases} 0 & \text{if } j \neq \dim_{\mathbb{C}}(G/L) \\ F^\lambda & \text{if } j = \dim_{\mathbb{C}}(G/L). \end{cases}$$

Schmid proved an analogous theorem about realizing discrete series. For Schmid's setting, G is a noncompact semisimple group, K is a maximal compact subgroup, and T is a maximal torus of K that is also maximal abelian in G. In this setting under the assumption that the parameter is dominant and very nonsingular, Schmid proved that $\bar\partial$ has closed image, that nonzero Dolbeault cohomology occurs only in degree $S = \dim_{\mathbb{C}}(K/T) = \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k})$, and that the smooth representation in degree S is infinitesimally equivalent with the expected discrete-series representation. Aguilar-Rodriguez extended Schmid's theorem to handle all discrete series.

Handling further cases of $H^{0,j}(G/L, V)$ presents formidable problems. One difficulty is in proving that $\bar\partial$ has closed image; this step was carried out for general G and finite-dimensional V by H.-W. Wong. Another difficulty is that $H^{0,j}(G/L, V)$ carries no obvious inner product. In parabolic induction, the inner products arise by integration, with the norm given by that for a vector-valued $L^2(K)$. However, $H^{0,j}(G/L, v)$ is a space of Dolbeault cohomology classes on a noncompact complex manifold. To construct an inner product analytically, one must show that the K finite cohomology classes have strongly harmonic representatives, face the fact that the L invariant Hermitian form on each fiber $(\bigwedge^j \mathfrak{u})^* \otimes_{\mathbb{C}} V$ may not be positive definite if L is noncompact, and prove that the strongly harmonic representatives of the K finite cohomology classes are square integrable on G/L. Except in isolated special cases chiefly in mathematical physics, the first progress in this direction was due to Rawnsley, Schmid, and Wolf, and came under various complex-analysis assumptions on G/L. Barchini, Knapp, and Zierau showed how to obtain strongly harmonic representatives with a mild real-analysis restriction on G/L, and Barchini was able to drop this restriction (retaining only the assumption of finite-dimensional fiber). Zierau has shown how in some cases square integrability on G/L may be deduced for the strongly harmonic representatives.

In any event, direct progress with the analytic setting has been slow in coming. Zuckerman's contribution, introduced in the next section, was to create an algebraic analog of this complex-analysis setting, thereby bypassing many of the analytic difficulties.

4. Zuckerman Functors

Zuckerman functors provide an algebraic analog of the complex-analysis construction in §3. They were introduced by Zuckerman in a series of lectures in 1978 and were developed further by Vogan [1981a]. For this section we use the following notation:

$$\begin{aligned}
&G = \text{linear connected reductive Lie group}\\
&K = \text{a maximal compact subgroup}\\
&T = \text{a torus in } G\\
&L = Z_G(T)\\
&Q = \text{parabolic subgroup in } G_{\mathbb{C}} \text{ as in §1}\\
&\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}\\
&(\sigma, V) = \text{smooth representation of } L.
\end{aligned} \tag{0.34}$$

The space V can be infinite dimensional, but we shall treat it as finite dimensional for the current purposes of motivation. The representation (σ, V) gives us a representation of $\mathfrak{l}$, and we extend this to a representation of $\mathfrak{q}$ by making $\mathfrak{u}$ act as 0. It will be helpful for purposes of motivation to think of the representation of $\mathfrak{q}$ on V as coming from a holomorphic representation of Q on V, but this assumption can be avoided.

In the analytic setting, $\bar{\partial}$ is an operator

$$\bar{\partial} : \mathcal{E}(G \times_L ((\wedge^j \mathfrak{u})^* \otimes V)) \to \mathcal{E}(G \times_L ((\wedge^{j+1} \mathfrak{u})^* \otimes V)). \tag{0.35}$$

Using the isomorphism (0.20), we regard $\bar{\partial}$ as an operator with domain equal to the space of smooth functions φ from G into $(\wedge^j \mathfrak{u})^* \otimes V$ satisfying

$$\varphi(gl) = (\mathrm{Ad}(l)^{-1} \otimes \sigma(l)^{-1})\varphi(g) \qquad \text{for } g \in G,\ l \in L \tag{0.36}$$

and with range equal to the corresponding space of functions into $(\wedge^{j+1} \mathfrak{u})^* \otimes V$.

In the algebraic analog we try to construct only the K finite vectors of $H^{0,j}$, thus obtaining a $(\mathfrak{g}, K)$ module. Let $\mathcal{C}(\mathfrak{g}, K)$ be the category of all $(\mathfrak{g}, K)$ modules.

The idea is to work with the Taylor coefficients at $g = 1$ of the function φ in (0.36), regarding each coefficient as attached to a left-invariant complex derivative (of some order) of φ at $g = 1$. Thus the idea of passing to Taylor coefficients gives us a linear map

$$\varphi \mapsto \varphi^{\#} \in \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{g}), (\wedge^j \mathfrak{u})^* \otimes V).$$

The transformation law (0.36) forces

(0.37) $$\varphi^\# \in \mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}),\ (\wedge^j \mathfrak{u})^* \otimes V),$$

where $\mathfrak{l}$ acts on $U(\mathfrak{g})$ on the right. If we assume that φ is K finite, then the action of $L \cap K$ on the left of φ gives an action of $L \cap K$ on $\varphi^\#$ by $\mathrm{Hom}(\mathrm{Ad},\ \mathrm{Ad}^* \otimes \sigma)$, and $\varphi^\#$ will be $L \cap K$ finite. Thus $\varphi^\#$ lies in a subspace that we denote

(0.38) $$\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}),\ (\wedge^j \mathfrak{u})^* \otimes V)_{L\cap K}$$

to indicate the $L \cap K$ finiteness. On (0.38) we have a representation of $\mathfrak{g}$ (via the action of $U(\mathfrak{g})$ on the left) and the representation of $L \cap K$, and (0.38) is a $(\mathfrak{g}, L \cap K)$ module.

The passage from the space of φ's as in (0.36) to the space of $\varphi^\#$'s in (0.38) loses information because

(a) φ need not be analytic, and hence $\varphi \mapsto \varphi^\#$ is not one-one
(b) formal power series do not have to converge and convergent power series do not have to globalize, and hence $\varphi \mapsto \varphi^\#$ is not onto.

We can get around the difficulties in (a) and (b) by defining away the problem. Let $\Gamma = \Gamma_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K}$ be the functor

$$\Gamma : \mathcal{C}(\mathfrak{g}, L \cap K) \to \mathcal{C}(\mathfrak{g}, K)$$

given by

$\Gamma(V) =$ sum of all finite-dimensional $\mathfrak{k}$ invariant subspaces of V for which the action of $\mathfrak{k}$ globalizes to K,

$\Gamma(\psi) = \psi|_{\Gamma(V)}$ if $\psi \in \mathrm{Hom}(V, W)$.

The functor Γ is covariant and left exact and is called the **Zuckerman functor**.

IDEA. Impose $\bar\partial$ between spaces

(0.39) $$\Gamma(\mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}),\ (\wedge^j \mathfrak{u})^* \otimes V)_{L\cap K}),$$

and take the kernel/image as a $(\mathfrak{g}, K)$ module analog of $H^{0,j}(G/L, V)$.

Let us bring in homological algebra, temporarily assuming that $L \subseteq K$. Then we make the following observations:

1) For the case $j = 0$ at least when V is finite dimensional, the condition that $\bar\partial\varphi^{\#} = 0$ is that $Z\varphi = 0$ for all $Z \in \mathfrak{u}$, in view of (0.22b). Thus the kernel/image space for $j = 0$ should be regarded as

$$\Gamma(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K}). \tag{0.40}$$

2) Identification of (0.40) as the space of interest for $j = 0$ suggests looking at the sequence

(0.41)
$$0 \longrightarrow \mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K} \longrightarrow \mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}),\ (\wedge^0\mathfrak{u})^* \otimes V)_{L\cap K}$$
$$\longrightarrow \mathrm{Hom}_{\mathfrak{l}}(U(\mathfrak{g}),\ (\wedge^1\mathfrak{u})^* \otimes V)_{L\cap K} \longrightarrow \cdots$$

in the category $\mathcal{C}(\mathfrak{g}, L\cap K)$. In fact, it can be proved that (0.41) is an injective resolution of $\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K}$ in the category $\mathcal{C}(\mathfrak{g}, L\cap K)$.

3) The category $\mathcal{C}(\mathfrak{g}, L\cap K)$ has enough injectives. Combining (2) and the idea above about (0.39), we see that the j^{th} space of interest, namely the j^{th} kernel/image of (0.39), is

$$\Gamma^j(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K}), \tag{0.42}$$

where Γ^j is the j^{th} right derived functor of Γ. (In fact, (0.42) is defined as the j^{th} cohomology of the complex obtained by applying Γ to (0.41), since (0.41) is an injective resolution.)

4) The space (0.42) gives the underlying $(\mathfrak{g}, K)$ module of K finite vectors of $H^{0,j}(G/L, V)$ for the cases of compact groups and the discrete series. These results are due essentially to Zuckerman and are proved in Vogan [1981a].

These observations lead us to the second crucial idea.

IDEA. Even when L is not compact, *define* the j^{th} space of interest to be (0.42).

In short, the Zuckerman construction is to pass from V in $\mathcal{C}(\mathfrak{l}, L\cap K)$ first to $\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K}$ in $\mathcal{C}(\mathfrak{g}, L\cap K)$ and then to $\Gamma^j(\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K})$ in $\mathcal{C}(\mathfrak{g}, K)$.

5. Cohomological Induction

Let G be connected semisimple with finite center. In keeping with the ideas of §4, we call

$$\mathcal{R}^j(Z) = \Gamma^j(\mathrm{pro}_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}(Z^\#))$$

a **cohomological induction** functor. Here

$$Z^\# = Z \otimes_{\mathbb{C}} \textstyle\bigwedge^{\mathrm{top}}\mathfrak{u} \quad \text{and} \quad \mathrm{pro}_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}(V) = \mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), V)_{L\cap K}.$$

The passage $Z \mapsto Z^\#$ is a normalization included to be consistent with the notation in the third form of the Bott-Borel-Weil Theorem (Theorem 0.33), and the compositions $\mathcal{R}^j$ carry $\mathcal{C}(\mathfrak{l}, L \cap K)$ to $\mathcal{C}(\mathfrak{g}, K)$.

What §4 shows is that the functors $\mathcal{R}^j$ provide a reasonable algebraic analog of the Dolbeault cohomology functors $H^{0,j}(G/L, Z^\#)$. In order to discuss unitarity, we need to see how these functors affect Hermitian forms. Here we find an unpleasant surprise: $\mathcal{R}^j$ cannot be applied naturally to Hermitian forms. Roughly speaking, the problem is that

$$\mathrm{pro}_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}(Z^\#) = \mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), Z^\#)_{L\cap K} \cong \mathrm{Hom}_{\mathbb{C}}(U(\bar{\mathfrak{u}}), Z^\#)_{L\cap K}$$

is simply too large to carry such a form. (Actually the imposed $L \cap K$ finiteness allows one to find invariant Hermitian forms on $\mathrm{pro}_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}(Z^\#)$, but not in any natural way.) The only consolation is that the Dolbeault cohomology has a parallel problem: For Z finite-dimensional, it follows from the work of Wong that $H^{0,j}(G/L, Z^\#)$ can carry an invariant Hermitian form only when the cohomology is finite dimensional. The forms arising in Schmid's construction of the discrete series, for example, are defined only on certain dense subspaces of cohomology.

So we start over. Suppose again that Z is an $(\mathfrak{l}, L \cap K)$ module. The first step is to regard $Z^\#$ as a $(\bar{\mathfrak{q}}, L \cap K)$ module on which $\bar{\mathfrak{u}}$ acts by 0. The second step is to apply an "algebraic induction" functor to form a $(\mathfrak{g}, L \cap K)$ module

$$\mathrm{ind}_{\bar{\mathfrak{q}},L\cap K}^{\mathfrak{g},L\cap K}(Z^\#) = U(\mathfrak{g}) \otimes_{\bar{\mathfrak{q}}} Z^\# \cong U(\mathfrak{u}) \otimes_{\mathbb{C}} Z^\#.$$

The third step is to apply some projective version $\Pi_j = (\Pi_{\mathfrak{g},L\cap K}^{\mathfrak{g},K})_j$ of Γ^j to get a $(\mathfrak{g}, K)$ module:

$$\mathcal{L}_j(Z) = \Pi_j(\mathrm{ind}_{\bar{\mathfrak{q}},L\cap K}^{\mathfrak{g},L\cap K}(Z^\#)).$$

We refer to $\mathcal{L}_j$ also as a **cohomological induction** functor.

The geometric setting of §3 does not suggest what Π should be. Instead we look for a direct algebraic definition, aiming to have many maps associated with Π go in the opposite direction of maps for Γ and to have Π be right exact rather than left exact. With this goal in mind, Π should be related to "largest K finite quotients" in the same way that Γ is related to "largest K finite subspaces." But largest K finite quotients do not always exist, and the definition of Π takes a little care. The first rigorous definition of Π was given by Bernstein in 1983. Let us postpone discussion of what is necessary to the next section. Historically the original attacks on unitarizability took Theorem 0.44a below as a definition of Π and its derived functors, and we can use this somewhat unsatisfactory approach as an interim measure.

An invariant sesquilinear form on a module V arises from a map of V into its Hermitian dual V^h. (See §VI.2 below for the definition of V^h.) To carry an invariant Hermitian form from Z to $\mathcal{L}_j(Z)$, we need a procedure for passing from a map $Z \to Z^h$ to a map $\mathcal{L}_j(Z) \to [\mathcal{L}_j(Z)]^h$. We give this procedure one step at a time. In our three-step construction, the map $Z \to Z^h$ easily gives a map from $Z^\#$ to $[Z^\#]^h$ and then a map from the $(\bar{\mathfrak{q}}, L\cap K)$ module Z to the $(\mathfrak{q}, L\cap K)$ module $[Z^\#]^h$. The second and third steps are handled by the proposition and theorem that follow.

We say that the $(\mathfrak{l}, L\cap K)$ module Z has **finite length** if Z has a (finite) composition series whose quotients are irreducible. In this case, any irreducible representation of the compact group $L\cap K$ occurs in Z with only finite multiplicity (see Theorem 10.1 below).

Proposition 0.43. Suppose Z is an $(\mathfrak{l}, L\cap K)$ module. Then

(a) there is a natural $(\mathfrak{g}, L\cap K)$ map

$$\operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^\#) \to \operatorname{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^\#)$$

that is nonzero if Z is nonzero,

(b) there is a natural isomorphism

$$[\operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^\#)]^h \cong \operatorname{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K}([Z^\#]^h),$$

(c) any nonzero invariant Hermitian form $\langle \cdot, \cdot \rangle_L$ on Z induces a nonzero invariant Hermitian form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^\#)$.

This proposition is elementary and is addressed at the beginning of §VI.4 below. In particular, the composition of the map in (a), followed by pro of the map $Z^\# \to [Z^\#]^h$ and then the inverse of the map in (b), carries $\operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^\#)$ to its Hermitian dual and defines the form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ in (c).

Theorem 0.44. Let $S = \dim \mathfrak{u} \cap \mathfrak{k}$. Then

(a) there is a natural isomorphism of functors

$$(\Pi_{\mathfrak{g},L\cap K}^{\mathfrak{g},K})_j \cong (\Gamma_{\mathfrak{g},L\cap K}^{\mathfrak{g},K})^{2S-j} \qquad \text{for } 0 \le j \le 2S,$$

(b) for $W \in \mathcal{C}(\mathfrak{g}, L \cap K)$, there is a natural isomorphism

$$[\Pi_j(W)]^h \cong \Gamma^j(W^h) \qquad \text{for } 0 \le j \le 2S,$$

(c) any invariant Hermitian form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on a $(\mathfrak{g}, L \cap K)$ module W induces an invariant Hermitian form $\langle \cdot, \cdot \rangle_G$ on $\Pi_S(W)$.

In the approach that we shall take in this book, parts (a) and (b) are substantially the Duality Theorem, the first main theorem of the book, which is proved in Chapter III below. Part (a) is most of Hard Duality, and part (b) is an instance of Easy Duality. Part (c) is then a formal consequence. If, as an interim measure as suggested above, (a) is taken as a definition of Π and its derived functors, then (b) is substantially the Duality Theorem in its original form as stated by Zuckerman and Enright-Wallach [1980].

Corollary 0.45. If Z is an $(\mathfrak{l}, L \cap K)$ module of finite length, then an invariant Hermitian form $\langle \cdot, \cdot \rangle_L$ on Z induces an invariant Hermitian form $\langle \cdot, \cdot \rangle_G$ on $\mathcal{L}_S(Z)$.

Recall that we have been seeking a complex-analysis construction (or an algebraic analog of one) yielding irreducible unitary representations and complementing the real-analysis construction of parabolic induction. We intend for cohomological induction with $\mathcal{L}_S$ to be that construction. Before considering how close we are to the desired goal, we mention one more theorem as background.

Theorem 0.46. If Z is an $(\mathfrak{l}, L \cap K)$ module of finite length, then

(a) $\operatorname{ind}_{\bar{\mathfrak{q}},L\cap K}^{\mathfrak{g},L\cap K}(Z^\#)$ and $\operatorname{pro}_{\mathfrak{q},L\cap K}^{\mathfrak{g},L\cap K}(Z^\#)$ have finite length, and they have the same irreducible composition factors and multiplicities

(b) all the $(\mathfrak{g}, K)$ modules $\mathcal{L}_j(Z)$ and $\mathcal{R}^j(Z)$ have finite length, and

$$\sum_j (-1)^j(\mathcal{L}_j(Z)) = \sum_j (-1)^j(\mathcal{R}^j(Z))$$

in the Grothendieck group of finite-length $(\mathfrak{g}, K)$ modules.

Part (a) is proved in §V.2 and §V.7 under a positivity hypothesis on Z, and the general case may be deduced from the special case by an argument with tensor products. The conclusion of finite length in (b) is proved in §V.2 and §V.4, and the identity in the Grothendieck group follows from part (a), Theorem 0.44a, and the long exact sequences for the derived functors of Π and Γ.

The discussion before Theorem 0.33 suggests aiming for interesting $(\mathfrak{g}, K)$ modules to occur as $\mathcal{R}^j(Z)$ with $j = S$, and Corollary 0.45 suggests that the $(\mathfrak{g}, K)$ module to consider for unitarity is $\mathcal{L}_j(Z)$ with $j = S$. Referring to Theorem 0.46, we see a way for $\mathcal{L}_S(Z)$ to match $\mathcal{R}^S(Z)$, namely that they be irreducible and that $\mathcal{L}_j(Z) = \mathcal{R}^j(Z) = 0$ for $j \neq S$. We are thus led to consider the following two problems.

PROBLEM A. Under what conditions can we conclude that $\mathcal{L}_j(Z)$ and $\mathcal{R}^j(Z)$ are 0 for $j \neq S$ and that $\mathcal{L}_S(Z)$ is irreducible?

PROBLEM B. When is $\mathcal{L}_S(Z)$ infinitesimally unitary?

For the most part, we shall need to assume some positivity condition on Z in order to make much progress. But there is one thing that can be said without assuming any positivity condition. Starting from the $(\mathfrak{l}, L \cap K)$ module Z, we can forget part of the action and regard Z as an $(\mathfrak{l} \cap \mathfrak{k}, L \cap K)$ module. The cohomological induction functor $\mathcal{L}_S$ for G has an analog for K given by

$$\mathcal{L}_S^K(Z) = (\Pi_{\mathfrak{k}, L\cap K}^{\mathfrak{k}, K})_S(\operatorname{ind}_{\bar{\mathfrak{q}}\cap\mathfrak{k}, L\cap K}^{\mathfrak{k}, L\cap K}(Z^\#)),$$

and this operates summand by summand on the irreducible constituents of the fully reducible $(\mathfrak{l}\cap\mathfrak{k}, L\cap K)$ module Z. A version of the third form of the Bott-Borel-Weil Theorem (Theorem 0.33) for $\mathcal{L}_S^K$ shows that an $L\cap K$ irreducible constituent of Z maps to an irreducible K representation or 0 depending on whether a certain translate of its highest weight is dominant for K. A K type (i.e., an equivalence class of irreducible representations of K) is said to be in the **bottom layer** if it occurs in $\mathcal{L}_S^K(Z)$. In §V.6 below it is shown that the **bottom-layer map**

$$\mathcal{B} : \mathcal{L}_S^K(Z) \to \mathcal{L}_S(Z) \qquad \text{given by} \qquad \Pi_S \circ \text{inclusion}$$

is one-one onto the full K isotypic subspaces for the K types of the bottom layer in $\mathcal{L}_S(Z)$.

The second main theorem of the book is the Signature Theorem. A special case of it says that if $\langle \cdot, \cdot \rangle_L$ is positive definite on the $L \cap K$

types of Z for which $\mathcal{L}_S^K$ is nonzero, then $\langle \cdot, \cdot \rangle_G$ is positive definite on the K types of the bottom layer in $\mathcal{L}_S(Z)$. More generally it says that an invariant notion of signature is preserved in passing from these $L \cap K$ types of Z to the K types of the bottom layer in $\mathcal{L}_S(Z)$.

6. Hecke Algebra and the Definition of Π

The beginnings of a definition of Π date back to Zuckerman's 1978 lectures and to ideas proposed at the time by Trauber and Borel. In connection with a possible proof of Hard Duality for Γ, Trauber and Borel suggested introducing the complex convolution algebra $R(\mathfrak{g}, K)$ of all left and right K finite distributions on G with support in K. In the same way that $\mathfrak{g}$ modules amount to the same thing as left $U(\mathfrak{g})$ modules in which 1 acts as 1, $(\mathfrak{g}, K)$ modules are identified with certain $R(\mathfrak{g}, K)$ modules. The algebra $R(\mathfrak{g}, K)$ usually does not have an identity, only an "approximate identity," and the condition that 1 act as 1 should be replaced by the condition "approximately unital," i.e., that, on each element of the module, members far out in the approximate identity act as 1. With this definition, $(\mathfrak{g}, K)$ modules amount to the same thing as left $R(\mathfrak{g}, K)$ modules that are approximately unital. (See §I.4 below.)

We call $R(\mathfrak{g}, K)$ the **Hecke algebra** for $(\mathfrak{g}, K)$. As is shown in Proposition 2.70 below, the functor $\Gamma = \Gamma_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K}$ is then given by

$$\Gamma(W) = \mathrm{Hom}_{R(\mathfrak{g}, L\cap K)}(R(\mathfrak{g}, K), V)_K.$$

In the language of homological algebra of rings and modules, Γ is a Hom type change-of-rings functor (except for the condition of K finiteness carried in the subscript K). The theory of change-of-rings functors suggests looking also at the corresponding tensor-product type functor, and this we may take as Π:

$$\Pi(W) = R(\mathfrak{g}, K) \otimes_{R(\mathfrak{g}, L\cap K)} W.$$

This is the definition that was used in Knapp-Vogan [1986]. Bernstein [1983] had earlier introduced an equivalent definition of Π in the course of investigating the correspondence between two different classifications of irreducible $(\mathfrak{g}, K)$ modules, and consequently we call Π the **Bernstein functor**.

The definitions of ind and pro as

$$\mathrm{ind}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, K}(V) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} V$$

and

$$\mathrm{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, K}(V) = \mathrm{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), V)_{L\cap K}$$

appear to be further changes of rings, at first glance from $U(\mathfrak{q})$ or $U(\bar{\mathfrak{q}})$ to $U(\mathfrak{g})$. But use of $U(\mathfrak{q})$, $U(\bar{\mathfrak{q}})$, and $U(\mathfrak{g})$ as the rings ignores the operation of $L \cap K$. The changes of rings should be from the rings appropriate to $(\mathfrak{q}, L \cap K)$ or $(\bar{\mathfrak{q}}, L \cap K)$ modules to the algebra $R(\mathfrak{g}, L \cap K)$, which is appropriate for $(\mathfrak{g}, L \cap K)$ modules.

Here we encounter a complication. The definition of $R(\mathfrak{g}, K)$ in terms of distributions assumed that $\mathfrak{g}$ is the complexification of the Lie algebra $\mathfrak{g}_0$ of a group G in which K is a subgroup, and $(\mathfrak{q}, L \cap K)$ and $(\bar{\mathfrak{q}}, L \cap K)$ do not fit this description. Thus we cannot immediately define $R(\mathfrak{q}, L \cap K)$ and $R(\bar{\mathfrak{q}}, L \cap K)$ in terms of distributions. Of course, we could attempt a definition of $R(\mathfrak{q}, L \cap K)$ and $R(\bar{\mathfrak{q}}, L \cap K)$ as subalgebras of $R(\mathfrak{g}, L \cap K)$, but fixing a total $\mathfrak{g}$ in which to operate would surely result in trouble eventually.

Thus what is needed is an algebraic construction of $R(\mathfrak{g}, K)$. Early joint work of Knapp and Vogan on such a construction appears in Knapp [1988]. By separating the parallel and transverse parts of the distributions that appear in $R(\mathfrak{g}, K)$, we show in §I.4 that

(0.47a) $$R(\mathfrak{g}, K) \cong R(K) \otimes_{U(\mathfrak{k})} U(\mathfrak{g}),$$

where $R(K)$ denotes the algebra of left and right K finite distributions on K (which are simply the K finite functions times Haar measure). The trouble with the isomorphism (0.47a) is that the multiplication law is lost. By separating parts in the reverse order, however, we obtain a second isomorphism

(0.47b) $$R(\mathfrak{g}, K) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} R(K).$$

Understanding the relationship between (0.47a) and (0.47b) leads to the multiplication rule, which can then be used to define an abstract version of $R(\mathfrak{g}, K)$.

Chapter I below gives a version of this algebraic construction that improves on what is in Knapp [1988]. With the construction in place, Chapter II takes up the question of change of rings. In an expression

$$\Pi^{\mathfrak{g},K}_{\mathfrak{g},L\cap K}(\operatorname{ind}^{\mathfrak{g},L\cap K}_{\bar{\mathfrak{q}},L\cap K}(V)) \qquad \text{with } V \in \mathcal{C}(\bar{\mathfrak{q}}, L \cap K),$$

both operations Π and ind are changes of rings, first from $R(\bar{\mathfrak{q}}, L \cap K)$ to $R(\mathfrak{g}, L \cap K)$ and then from $R(\mathfrak{g}, L \cap K)$ to $R(\mathfrak{g}, K)$. They can therefore be telescoped into a single change from $R(\bar{\mathfrak{q}}, L \cap K)$ to $R(\mathfrak{g}, K)$:

$$P^{\mathfrak{g},K}_{\bar{\mathfrak{q}},L\cap K}(V) = R(\mathfrak{g}, K) \otimes_{R(\bar{\mathfrak{q}},L\cap K)} V.$$

Similar remarks apply to Γ and pro. The one-step Hom type change-of-rings functor is

$$I_{\mathfrak{q},L\cap K}^{\mathfrak{g},K}(V) = \mathrm{Hom}_{R(\bar{\mathfrak{q}},L\cap K)}(R(\mathfrak{g},K),V)_K.$$

More generally we see that $P_{\mathfrak{h},B}^{\mathfrak{g},K}$ and $I_{\mathfrak{h},B}^{\mathfrak{g},K}$ make sense whenever $\mathfrak{h} \subseteq \mathfrak{g}$ and $B \subseteq K$ compatibly. In fact, the inclusions can be replaced by maps $i_{\mathrm{alg}} : \mathfrak{h} \to \mathfrak{g}$ and $i_{\mathrm{gp}} : B \to K$ with suitable compatibility properties. The extended definitions of the functors P and I for this situation are

$$P_{\mathfrak{h},B}^{\mathfrak{g},K}(V) = R(\mathfrak{g},K) \otimes_{R(\mathfrak{h},B)} V$$

and

$$I_{\mathfrak{h},B}^{\mathfrak{g},K}(V) = \mathrm{Hom}_{R(\mathfrak{h},B)}(R(\mathfrak{g},K),V)_K.$$

Chapter II develops the theory in this generality. The functors P and I will have as special cases Π and Γ, ind and pro, $\Pi \circ \mathrm{ind}$ and $\Gamma \circ \mathrm{pro}$, and coinvariants and invariants. The derived functors of P and I will have as special cases Π_j and Γ^j, $(\Pi \circ \mathrm{ind})_j \cong \Pi_j \circ \mathrm{ind}$ and $(\Gamma \circ \mathrm{pro})^j \cong \Gamma^j \circ \mathrm{pro}$, and Lie algebra homology and cohomology. Thus P and I, along with their derived functors, are pervasive in the theory.

7. Positivity and the Good Range

Let us return to Problems A and B in §5. Again G is a connected semisimple Lie group with finite center, K is a maximal compact subgroup, and $\mathfrak{g}$ is the complexified Lie algebra of G. As mentioned, we need to assume some positivity condition on the $(\mathfrak{l}, L \cap K)$ module Z in order to make much progress on the two problems.

At the same time that Harish-Chandra was introducing Verma modules (see §1), he investigated the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$. Let $\mathfrak{h}$ be any Cartan subalgebra of $\mathfrak{g}$. Harish-Chandra introduced a map $\gamma_{\mathfrak{g}}$ from $Z(\mathfrak{g})$ into the symmetric algebra $S(\mathfrak{h})$ and showed that $\gamma_{\mathfrak{g}}$ is an algebra isomorphism of $Z(\mathfrak{g})$ onto the algebra of Weyl-group invariants in $S(\mathfrak{h})$. (See §IV.7 below.) In terms of this map he proved that every homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is of the form $\chi = \chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$, where

$$\chi_\lambda(z) = \lambda(\gamma_{\mathfrak{g}}(z)).$$

Moreover, $\chi_\lambda = \chi_{\lambda'}$ if and only if λ and λ' are in the same orbit of the Weyl group. (See §IV.8 below.)

We say that a $U(\mathfrak{g})$ module V has **infinitesimal character** λ if $Z(\mathfrak{g})$ operates by scalars in V and if the homomorphism $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ defined by those scalars is $\chi = \chi_\lambda$. For any irreducible $U(\mathfrak{g})$ module V, a version of Schur's Lemma due to Dixmier (Proposition 4.87 below) says that V has an infinitesimal character.

For our situation with L and G, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{l}$. Then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$, and infinitesimal characters for $\mathfrak{l}$ and $\mathfrak{g}$ can both be given as members of $\mathfrak{h}^*$. We shall assume from now on that our $(\mathfrak{l}, L \cap K)$ module Z has an infinitesimal character, as well as finite length.

Let us pause for some examples. Let $\Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots of $\mathfrak{g}$, and let $\Delta(\mathfrak{u})$ and $\Delta(\mathfrak{l}, \mathfrak{h})$ denote the subsets of roots whose root vectors lie in $\mathfrak{u}$ and $\mathfrak{l}$, respectively. If we introduce a positive system $\Delta^+(\mathfrak{l}, \mathfrak{h})$ for $\mathfrak{l}$, then we can take $\Delta(\mathfrak{u}) \cup \Delta^+(\mathfrak{l}, \mathfrak{h})$ as a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ for $\mathfrak{g}$. Let δ_L, $\delta(\mathfrak{u})$, and δ be half the sum of the members of $\Delta^+(\mathfrak{l}, \mathfrak{h})$, $\Delta(\mathfrak{u})$, and $\Delta^+(\mathfrak{g}, \mathfrak{h})$, respectively. Note that $\delta = \delta_L + \delta(\mathfrak{u})$. If Z is an irreducible finite-dimensional $\mathfrak{l}$ module with highest weight μ, then Z has infinitesimal character $\mu + \delta_L$. The unique weight of $\bigwedge^{\text{top}}\mathfrak{u}$ is $2\delta(\mathfrak{u})$, and thus, in this case, $Z^\#$ has highest weight $\mu + 2\delta(\mathfrak{u})$ and infinitesimal character $\mu + \delta + \delta(\mathfrak{u})$. The following proposition is proved below in §V.2.

Proposition 0.48. If the $(\mathfrak{l}, L\cap K)$ module Z has infinitesimal character λ, then $Z^\#$ has infinitesimal character $\lambda + 2\delta(\mathfrak{u})$, while

$$\operatorname{ind}_{\bar{\mathfrak{q}}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^\#), \quad \operatorname{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^\#), \quad \mathcal{L}_j(Z), \quad \text{and} \quad \mathcal{R}^j(Z)$$

all have infinitesimal character $\lambda + \delta(\mathfrak{u})$.

In order to formulate positivity conditions, let $\langle \cdot\,, \cdot \rangle$ denote the Killing form on $\mathfrak{g}_0$. More generally, if $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the Cartan decomposition of $\mathfrak{g}_0$ relative to $\mathfrak{k}_0$, we can use as $\langle \cdot\,, \cdot \rangle$ any $\operatorname{Ad}(G)$ invariant nondegenerate symmetric bilinear form on $\mathfrak{g}_0$ that is negative definite on $\mathfrak{k}_0$, is positive definite on $\mathfrak{p}_0$, and has $\mathfrak{k}_0$ orthogonal to $\mathfrak{p}_0$. This form extends by complexification to all of $\mathfrak{g}$, by restriction to nondegenerate forms on both $\mathfrak{l}$ and $\mathfrak{h}$, and by dualization to $\mathfrak{h}^*$. The form is positive definite on the real span of the roots in $\mathfrak{h}^*$. For purposes of this Introduction, we make the following definition.

DEFINITION 0.49. With $\langle \cdot\,, \cdot \rangle$ as above, suppose that the $(\mathfrak{l}, L \cap K)$ module Z has an infinitesimal character λ. We say that Z or λ is in the **good range** or that Z is **good** (relative to $\mathfrak{q}$ and $\mathfrak{g}$) if

$$\operatorname{Re}\langle \lambda + \delta(\mathfrak{u}), \alpha \rangle > 0 \qquad \text{for all } \alpha \in \Delta(\mathfrak{u}).$$

We say that Z or λ is **weakly good** if

$$\mathrm{Re}\langle \lambda + \delta(\mathfrak{u}), \alpha \rangle \geq 0 \qquad \text{for all } \alpha \in \Delta(\mathfrak{u}).$$

These definitions are independent of the choice of the form $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}_0$, of the choice of Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$, and of the choice of λ from within its orbit under the Weyl group of $\mathfrak{l}$. A first answer to Problem A in §5 is as follows.

Theorem 0.50. Let Z be an $(\mathfrak{l}, L \cap K)$ module of finite length with infinitesimal character λ, and suppose that Z is weakly good. Then

(a) $\mathcal{L}_j(Z) = \mathcal{R}^j(Z) = 0$ for $j \neq S$
(b) $\mathcal{L}_S(Z) \cong \mathcal{R}^S(Z)$
(c) Z irreducible implies $\mathcal{L}_S(Z)$ is irreducible or zero.

If Z is assumed actually to be good, then (c) can be strengthened to

(c′) Z irreducible implies $\mathcal{L}_S(Z)$ is irreducible.

Parts (a) and (b) are given as a vanishing theorem in §V.7. Parts (c) and (c′) are essentially the Irreducibility Theorem (Theorem 8.2 below), the third main theorem of the book.

The need for the hypotheses in Theorem 0.50 can be understood already in the compact case. When G is compact and Z is not good, $\mathcal{L}_S(Z)$ is zero. If Z is not good and no root is orthogonal to $\lambda + \delta(\mathfrak{u})$, the vanishing result in (a) will fail.

A first answer to Problem B in §5 is as follows.

Theorem 0.51. Let Z be an $(\mathfrak{l}, L \cap K)$ module of finite length with infinitesimal character λ, let $\langle \cdot\,, \cdot \rangle_L$ be a nonzero invariant Hermitian form on Z, and let $\langle \cdot\,, \cdot \rangle_G$ be the corresponding invariant Hermitian form on $\mathcal{L}_S(Z)$. If Z is weakly good, then

(a) $\langle \cdot\,, \cdot \rangle_L$ nondegenerate implies $\langle \cdot\,, \cdot \rangle_G$ nondegenerate
(b) $\langle \cdot\,, \cdot \rangle_L$ positive definite implies $\langle \cdot\,, \cdot \rangle_G$ positive definite.

In particular, if Z is weakly good and Z is infinitesimally unitary, then $\mathcal{L}_S(Z)$ is infinitesimally unitary.

Part (a) is an observation in §VI.4 below. Part (b) is the Unitarizability Theorem (Theorem 9.1 below), the fourth main theorem of the book.

8. One-Dimensional Z and the Fair Range

We continue with the notation of §7. For special kinds of $(\mathfrak{l}, L \cap K)$ modules Z, some improvement is possible in Theorems 0.50 and 0.51. In this section we examine especially the case of one-dimensional Z. If λ' is the unique weight of Z, we write $Z = \mathbb{C}_{\lambda'}$. The infinitesimal character of $\mathbb{C}_{\lambda'}$ is $\lambda = \lambda' + \delta_L$, and the good range is given by $\langle \lambda + \delta(\mathfrak{u}), \alpha \rangle > 0$ for $\alpha \in \Delta(\mathfrak{u})$.

Let $\mathfrak{z}$ be the center of $\mathfrak{l}$. This is automatically a subspace of the Cartan subalgebra $\mathfrak{h}$.

DEFINITION 0.52. With $\langle \cdot, \cdot \rangle$ as above, suppose that the $(\mathfrak{l}, L \cap K)$ module Z has an infinitesimal character λ. We say that Z or λ is in the **fair range** or that Z is **fair** (relative to $\mathfrak{q}$ and $\mathfrak{g}$) if

$$\mathrm{Re}\langle \lambda + \delta(\mathfrak{u}), \alpha|_{\mathfrak{z}} \rangle > 0 \qquad \text{for all } \alpha \in \Delta(\mathfrak{u}).$$

We say that Z or λ is **weakly fair** if

$$\mathrm{Re}\langle \lambda + \delta(\mathfrak{u}), \alpha|_{\mathfrak{z}} \rangle \geq 0 \qquad \text{for all } \alpha \in \Delta(\mathfrak{u}).$$

When $Z = \mathbb{C}_{\lambda'}$ and $\lambda = \lambda' + \delta_L$, we have

$$\langle \lambda + \delta(\mathfrak{u}), \alpha|_{\mathfrak{z}} \rangle = \langle \lambda' + \delta(\mathfrak{u}), \alpha \rangle.$$

Thus the conditions "fair" and "weakly fair" say for all $\alpha \in \Delta(\mathfrak{u})$ that $\langle \lambda' + \delta(\mathfrak{u}), \alpha \rangle$ is >0 or ≥ 0, respectively.

Whether or not Z is one-dimensional, it is not hard to see that if Z is in the good range, then Z is in the fair range. Also if Z is in the weakly good range, then Z is in the weakly fair range.

Unfortunately the "fair" hypothesis does not imply analogs of Theorems 0.50 and 0.51 of §7 in general. But here is a first hint that there are positive results to be found.

Theorem 0.53. If Z is a weakly fair one-dimensional $(\mathfrak{l}, L \cap K)$ module with infinitesimal character λ, then

(a) $\mathcal{L}_j(Z) = \mathcal{R}^j(Z) = 0$ for $j \neq S$
(b) $\mathcal{L}_S(Z) \cong \mathcal{R}^S(Z)$
(c) the action of $U(\mathfrak{g})$ on $\mathcal{L}_S(Z)$ extends naturally to an algebra $D(G_{\mathbb{C}}/Q)_{(\lambda+\delta(\mathfrak{u}))|_{\mathfrak{z}}}$ of "twisted differential operators," and $\mathcal{L}_S(Z)$, as a $D(G_{\mathbb{C}}/Q)_{(\lambda+\delta(\mathfrak{u}))|_{\mathfrak{z}}}$ module, is irreducible or zero.

Parts (a) and (b) are in §V.7 below, along with parts (a) and (b) of Theorem 0.50. Discussion of (c) begins in §VIII.5 below and continues in Chapter XII. The definition of $D(G_{\mathbb{C}}/Q)_{(\lambda+\delta(\mathfrak{u}))|_{\mathfrak{z}}}$ will not concern us at this time. The point is that $U(\mathfrak{g})$ can be enlarged to a naturally defined algebra that always acts irreducibly on $\mathcal{L}_S(Z)$ if $\mathcal{L}_S(Z) \neq 0$. However, $U(\mathfrak{g})$ itself sometimes acts irreducibly and sometimes acts reducibly. Strengthening the hypothesis "weakly fair" to "fair" in Theorem 0.53 does not yield a conclusion (c) that is closer to (c) or (c′) of Theorem 0.50; for example, one cannot guarantee that $\mathcal{L}_S(Z)$ is nonzero, or that it is irreducible or zero as a $(\mathfrak{g}, K)$ module.

There is again a parallel result for unitarity.

Theorem 0.54. If Z is a weakly fair one-dimensional infinitesimally unitary $(\mathfrak{l}, L \cap K)$ module with invariant form $\langle \cdot, \cdot \rangle_L$, then the corresponding form $\langle \cdot, \cdot \rangle_G$ on $\mathcal{L}_S(Z)$ is positive definite, and consequently $\mathcal{L}_S(Z)$ is infinitesimally unitary.

It is natural to try to understand what it is about one-dimensional representations that makes Theorems 0.53 and 0.54 work. Doing so involves looking in detail at the proofs, and we postpone this project to Chapter XII. Examination of the proof of Theorem 0.54 leads to the definition of "weakly unipotent" $(\mathfrak{l}, L \cap K)$ modules. When such a module Z is weakly fair, we obtain the same conclusion as in Theorem 0.54, that Z infinitesimally unitary implies $\mathcal{L}_S(Z)$ infinitesimally unitary. The situation with generalizing Theorem 0.53 is more complicated. The algebra $D(G_{\mathbb{C}}/Q)_{(\lambda+\delta(\mathfrak{u}))|_{\mathfrak{z}}}$ leads to "Dixmier algebras," and irreducibility is expressed in terms of them. The algebra $U(\mathfrak{g})$ maps into a Dixmier algebra, with very large image, and a conclusion of irreducibility of $\mathcal{L}_S(Z)$ is valid when $U(\mathfrak{g})$ maps onto the Dixmier algebra. These matters are discussed in Chapter XII below.

9. Transfer Theorem

Now that we have a construction that often yields irreducible unitary representations, we want to be able to use them. In practice, being able to use these representations requires understanding how they fit into a classification and understanding how they can be constructed in other ways.

The first ingredient in this analysis is to realize parabolic induction on the level of $(\mathfrak{g}, K)$ modules. In parabolic induction we induce a smooth or

Hilbert-space representation from a parabolic subgroup of G to G itself. The usual convention is to start with an irreducible representation of M, a one-dimensional representation of A, and the trivial representation of N, and then to proceed as in §2. Translating the data by a certain nonunitary one-dimensional representation e^ρ on A ensures that unitary representations of MA lead to unitary representations of G. Let $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}$ be the complexified Lie algebra of MA, let $\mathfrak{n}$ be the complexified Lie algebra of N, and put $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$. If we write Z for the underlying $(\mathfrak{l}, L \cap K)$ module of the representation of MA and denote by $Z^\natural$ the effect of putting ρ in place, then the underlying $(\mathfrak{g}, K)$ module turns out to be

$$\Gamma_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K}(\operatorname{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K}(Z^\natural)).$$

This conclusion remains valid if Z is replaced by any $(\mathfrak{l}, L \cap K)$ module of finite length. Moreover

$$(\Gamma_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K})^j(\operatorname{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K})(Z^\natural)) = 0 \qquad \text{for } j > 0.$$

Thus the $(\mathfrak{g}, K)$ analog of parabolic induction is notationally similar to cohomological induction except on two points:

(a) the normalization $Z \mapsto Z^\natural$ is different from the earlier normalization $Z \mapsto Z^\#$

(b) the representation of interest occurs in cohomology of degree 0 rather than degree S.

Let us drop the normalizations for the remainder of this Introduction. (In practice, we eventually want some normalization back in place in order to make unitary representations go to unitary representations.) Suppose $\mathfrak{q}$ is any parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{q}^-$ is the opposite parabolic. Let us suppose that $\mathfrak{q} \cap \mathfrak{q}^- = \mathfrak{l}$ is the complexification of a real Lie subalgebra $\mathfrak{l}_0$ of $\mathfrak{g}_0$. We write $\mathfrak{u}$ for the nilpotent radical of $\mathfrak{q}$, so that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. By $L \cap K$ we mean any closed subgroup of K whose Lie algebra is $\mathfrak{l}_0 \cap \mathfrak{k}_0$ such that $\operatorname{Ad}(L \cap K)\mathfrak{u} \subseteq \mathfrak{u}$. Then we can form

$$(0.55) \qquad (\Gamma_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K})^j(\operatorname{pro}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K})(Z)) \quad \text{and} \quad (\Pi_{\mathfrak{g}, L\cap K}^{\mathfrak{g}, K})_j(\operatorname{ind}_{\mathfrak{q}, L\cap K}^{\mathfrak{g}, L\cap K})(Z)).$$

The problem is to understand the $(\mathfrak{g}, K)$ modules (0.55), relating them to each other as $\mathfrak{u}$ and j vary. There are two tools for doing so, and they can then be iterated:

(a) the Transfer Theorem addresses a one-step change in $\mathfrak{u}$ in the special case that $\mathfrak{l}$ reduces to a Cartan subalgebra. Under a condition on Z, the theorem matches the module in degree j

for one choice of $\mathfrak{u}$ with the module in degree $j+1$ for another choice of $\mathfrak{u}$.

(b) the double-induction spectral sequence addresses what happens when two Γ type constructions or two Π type constructions are composed.

These results are made precise and proven in Chapter XI. The Transfer Theorem is the fifth main theorem of the book.

The Transfer Theorem leads to striking relationships among $(\mathfrak{g}, K)$ modules (0.55) when L is a Cartan subgroup of G. Under some restrictions on Z, such a $(\mathfrak{g}, K)$ module is called **standard**. Various classification theorems are formulated in terms of quotients or submodules of standard modules. One such is the Langlands classification, which realizes irreducible representations as the result of a three-step process consisting of

(i) construction of discrete series and "limits of discrete series"
(ii) passage to a standard representation by Mackey induction
(iii) extraction of an irreducible quotient or subrepresentation (depending on the particular version of the Langlands classification, and depending on the use of Π or Γ in the classification).

Step (i) is given by cohomological induction, and step (ii) is parabolic induction. The Transfer Theorem implies that the same $(\mathfrak{g}, K)$ modules result if one goes through a three-step process consisting of

(i) construction of a "principal-series" representation of a group "split modulo center," using Mackey parabolic induction
(ii) passage to a standard representation by cohomological induction
(iii) extraction of an irreducible subrepresentation or quotient (depending on the use of Π or Γ in the classification).

One of the uses of these results is to place cohomologically induced modules in the Langlands classification, at least in the weakly good range. Using the results, one can transfer the Signature Theorem to a theorem cast solely in terms of the Langlands classification. The transferred theorem is a powerful tool for exhibiting Langlands parameters that do not correspond to unitary representations.